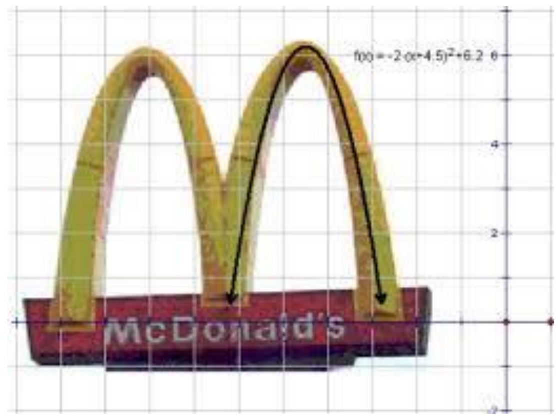
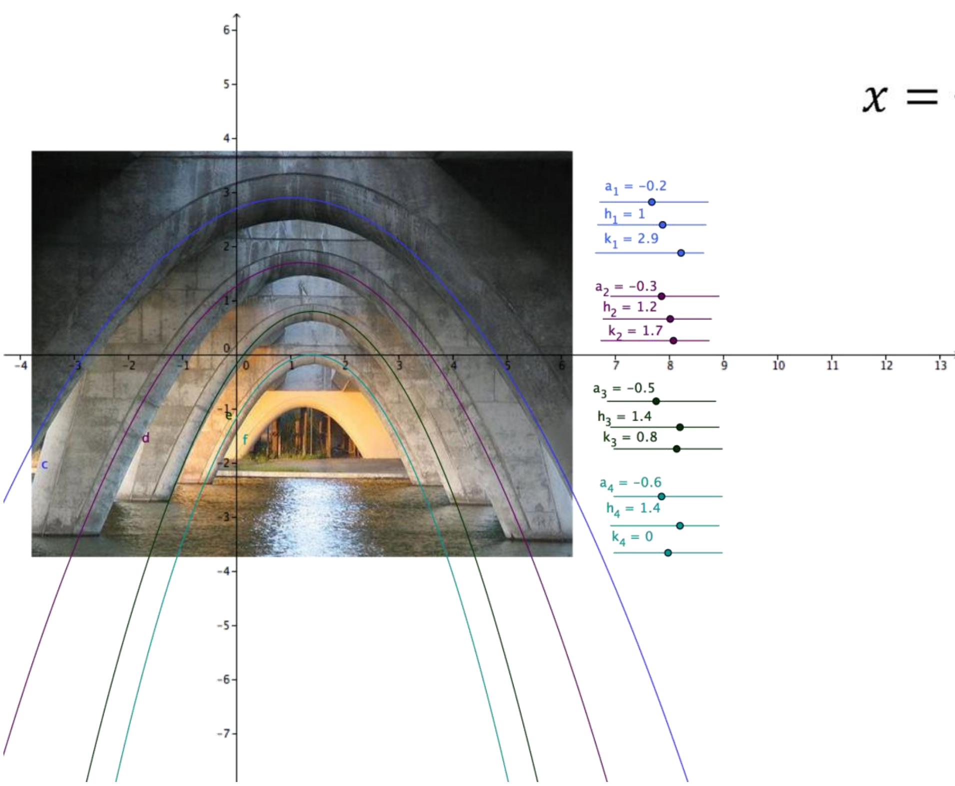
# Chapter 2






Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

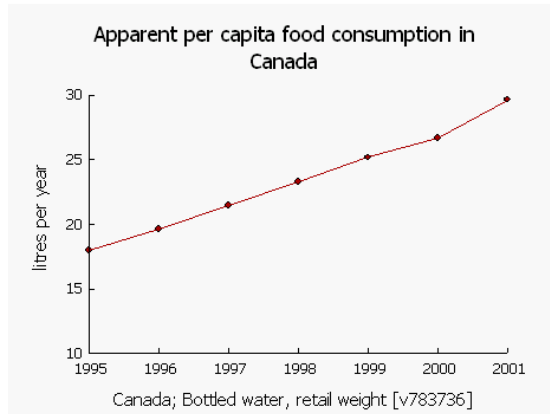


## 2.1 Tables, Graphs, and Equations

The graph of an equation is a visual representation of the solutions to the equation. A point is included on the graph if its coordinates satisfy the equation. Graphs can be used to show the relationship between two variables.

**Example:**

The graph shows the average amount of bottled water (in liters per year) consumed per person in Canada for the years from 1995 to 2001.




1. What is the average amount of bottled water consumed per person in 1996?

   There was approximately 20 liters per year consumed in 1996.

2. During what year was 25 liters per year consumed per person? 1999

A **complete graph** is a graph that shows all the key points and the behavior of the graph. Linear graphs should include the intercepts of the graph unless the intercept is not appropriate in the context of the problem.

The **intercepts** of a graph are the points where the graph intersects the axes. The horizontal intercept (or x-intercept) is the point where the graph crosses the x-axis. The vertical intercept (or y-intercept) is the point where the graph crosses the y-axis.

To find the intercepts:

1. The horizontal intercept is found by replacing the output variable (or y) with zero in the equation. Then solving for the remaining variable. You will have a point of the form (a, 0).

2. The vertical intercept is found by replacing the input variable (or x) with zero in the equation. Then solving for the remaining variable. You will have a point of the form (0, b).

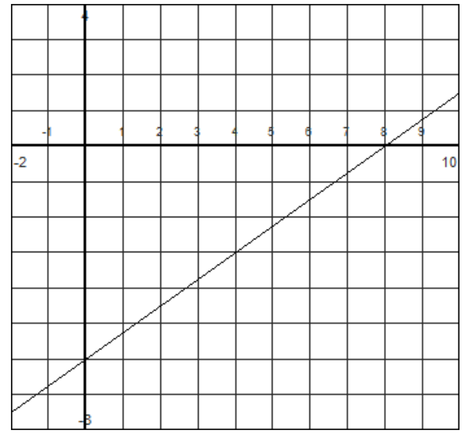
**Example:**

1. Find the intercepts of the equation $3x - 4y = 24$.

   A. To find the x-intercept, set $y = 0$.
      $3x - 4(0) = 24$
      $3x = 24$
      $x = 8$ so the x-intercept (horizontal intercept) is the point (8,0).

   B. To find the y-intercept, set $x = 0$.

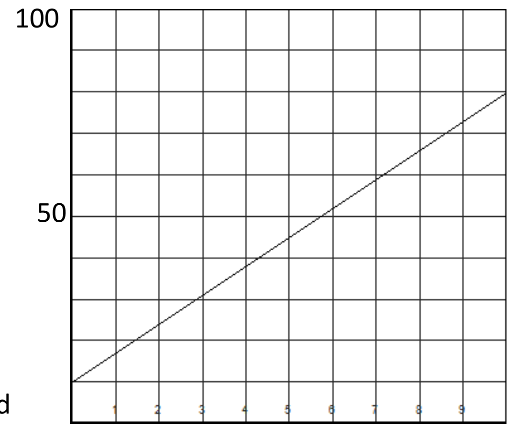
$3(0) - 4y = 24$
$-4y = 24$
$y = -6$ so the y-intercept (vertical intercept) is (0, -6).
A complete graph is shown to the right.



2. To order specialized M&Ms the cost is given by C = 9.99 + 6.99P where P is the number of pounds ordered and C is the total cost. Find the intercepts and discuss if they are reasonable in the context of the problem.

   A. To find the C-intercept, set P = 0.
      C = 9.99 + 6.99(0) = 9.99 so intercept is (0, 9.99)
      When 0 pounds of M&Ms are bought, the cost is $9.99.
      (This is the cost for shipping and handling).

   B. To find the P-intercept, set C = 0.
      0 = 9.99 + 6.99P
      -6.99P = 9.99
      P = -1.43 so intercept is (-1.43, 0)
      This would mean that if you bought a negative number of pounds of M&Ms then the cost would be $0. Negative pounds are not reasonable. This intercept would not need to be shown on the graph. A complete graph is shown.




There are four ways to model situations mathematically; these are numerical, algebraic, graphical, and verbal representations. If we use a table to represent our data, this is a **numerical representation**. If we use an equation to represent our information, then this is **an algebraic representation**. Using a graph is a **graphical representation** and words are a **verbal representation**. Each of these representations can be very useful in analyzing problem situations and solving problems. A table of values displays specific data points, a graph is a visual representation that can be used to identify trends and overall behavior, and equations can be used to analyze and make predictions.

**Using the graphing calculator to make a table:**

Step 1: Press [Y =] Input your equation (in y = form).

Step 2: Go to the table setup (TBLSET) menu. Decide whether you want the calculator to input the values or whether you want to choose individual values. If you want the calculator to input the values, choose a value for TblStart (the starting value) and ΔTbl (what the values are increased by) and set Indpnt to Auto. If you want to choose individual values for the table, then set Indpnt to Ask.

Step 3: Go to TABLE.

**Using the graphing calculator to make a graph:**

Step 1: Press [ Y = ] Input your equation (in y = form).

Step 2: Go to WINDOW. Enter an appropriate viewing window for the equation. This window should show a complete graph and the graph should be easy to read. Be sure to pick a scale for the axes that is appropriate to the values for the minimum and maximum values.

Step 3: Go to GRAPH.

**Examples:**

1. Sarah buys a $50 phone card when she travels abroad. To call home, calls cost 2.8 cents per minute.

A. Make a table of values showing the value of the card, V, for various lengths of time talked.

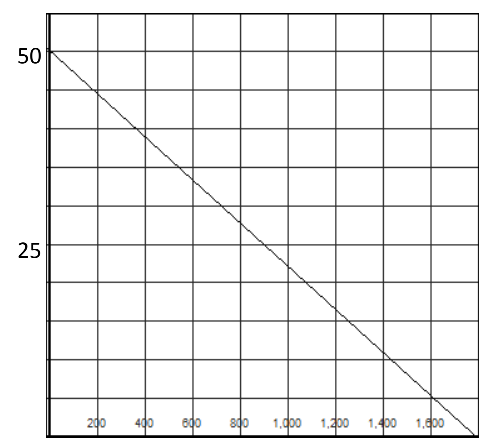
The card starts at an amount of $50 which will decrease by 2.8 cents for every minute Sarah talks on the phone. So the value will be $50 minus $0.028 times the number of minutes talked.

| Minutes talked, t | | Value of card in dollars, V |
|---|---|---|
| 0 | 50 – 0.028(0) | 50 |
| 100 | 50 – 0.028(100) | 47.20 |
| 500 | 50 – 0.028(500) | 36 |
| 1000 | 50 – 0.028(1000) | 22 |

B. Plot the points on a graph and draw a line through the data points. Be sure to show the intercepts on the graph if reasonable.

Using the points found above and some additional points, the graph is shown here.

| t | V |
|---|---|
| 0 | 50 |
| 500 | 36 |
| 1000 | 22 |
| 1500 | 8 |
| 1750 | 1 |




The vertical intercept is (0, 50) which represents the starting value of the card. The horizontal intercept is between 1750 and 1800 minutes. Negative values for time or the value of the card are not reasonable so the graph should be only shown in quadrant I.

C. Write an equation for V in terms of t.

Look back to the calculations in part A, to find the value of the card:
V = 50 – 0.028t, where t is in minutes and V is in dollars.

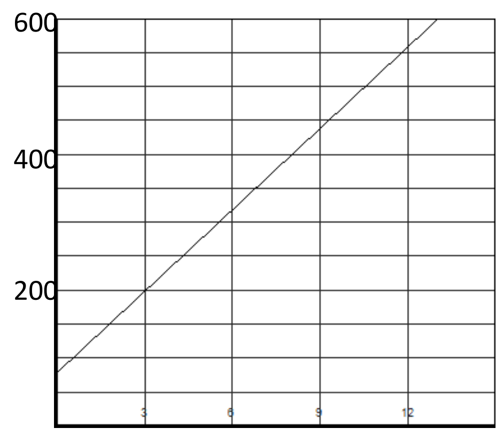
2. James is going to join a fitness club. The club has an initial fee of $79 and then a monthly fee of $39.99 per month.

A. Make a table of values showing the cost of his membership in terms of the number of months he uses the club.

James has an initial fee of $79 and then additional fees of $39.99 each month he belongs to the club. So, his total fees will be the initial fee plus his monthly fees.

| Months, M | | Total fees, F |
|---|---|---|
| 0 | 79 + 39.99(0) | 79 |
| 3 | 79 + 39.99(3) | 198.97 |
| 6 | 79 + 39.99(6) | 318.94 |
| 12 | 79 + 39.99(12) | 558.88 |

B. Plot the points on a graph and draw a line through the data points. Be sure to show the intercepts on the graph if reasonable.




The vertical intercept is at (0, 79) which represents the initial fee. The horizontal intercept would be at a negative number of months so it is not reasonable in the context of the problem. The graph should be only shown in quadrant I.

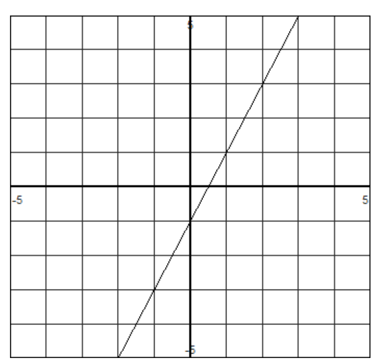
C. Write an equation for V in terms of t.

Look at the calculations in part A, to find the total fee:
F = 79 + 39.99M, where M is in months and F is in dollars.

**2.1 Homework:**

1. Use the graph to fill in the missing table values. Assume the scales are 1 unit.

| x | y |
|---|---|
| 0 | |
| | 3 |
| -2 | |
| | -3 |



Fill in the table of values for the given equation.

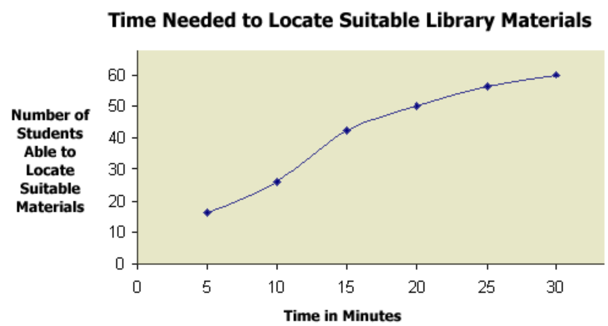
2. $\frac{1}{2}x - y = -4$

| x | y |
|---|---|
| 0 | |
| | 0 |
| -4 | |
| | 8 |

3. y = -2x + 1

| x | y |
|---|---|
| 0 | |
| | 0 |
| 4 | |
| | -9 |

4. Given the graph as shown, answer the following questions.

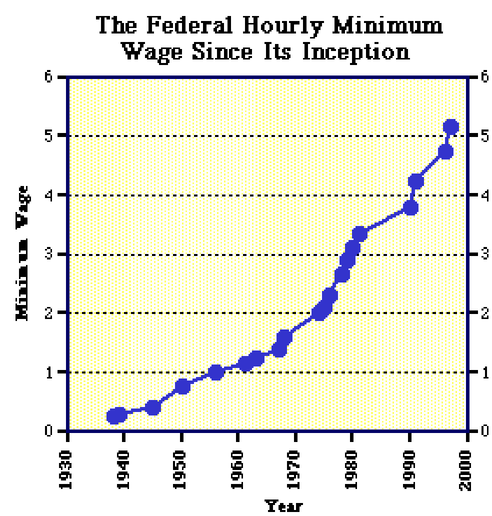



   A. Approximately how many students can find suitable materials in 20 minutes?

   B. What is the minimum time for at least 30 students to find suitable materials in the library?

   C. How many students could find materials in 10 minutes or more?

5. Given the graph as shown, answer the following questions.




   A. What was the approximate minimum wage in 1990?

   B. What was the approximate minimum wage in 1980?

   C. In what years did the minimum wage exceed $2?

   D. When did the minimum wage increase the fastest?

   E. Approximately how much did minimum wage rise during the 1960s?

6. Given 2x – 3y = 9:

   A. Solve the equation for y in terms of x.

   B. Find the horizontal intercept and vertical intercepts.

   C. Sketch the graph.

Find the intercepts of the following equations. Then, find an appropriate viewing window including the scale on the axes. Assume that the independent variable comes first alphabetically.

7. 2L + 3W = 90

8. $y = \frac{3}{4}x + 6$

9. 0.50C + 1.75P = 350

10. 200x + 350y = 20

11. There is a relationship between the outside temperature and the rate of travel (speed) of certain ants. Several observations were made (which are shown in the table below) where s is the speed in centimeters per second and t is the temperature in degrees Celsius.

| | | | | |
|---|---|---|---|---|
| temperature | 4 | 16 | 22 | 28 |
| speed | 0 | 2 | 3 | 4 |

A. What does the point (4, 0) represent in context?

B. Find an appropriate viewing window for the table values.

12. Given the equation: $y = \frac{2}{7}x - \frac{4}{5}$,

A. Fill in the table.

| **x** | 1 | 3 | 7 | 12 | 21 |
|---|---|---|---|---|---|
| **y** | | | | | |

B. Find an appropriate viewing window.

13. Cathy buys a 35-pound bag of rice and consumes 0.6 pounds per week.

A. Write an expression for the amount of rice that Cathy has left in terms of the number of weeks since she has bought the bag.

B. Find the amount of rice Cathy has left after 9 weeks.

C. Find the horizontal intercept and explain what it means in the context of this problem.

D. Find the vertical intercept and explain what it means in the context of this problem.

E. Sketch a complete graph of this equation.

14. Jan buys a car. The value of her car is given by the equation V = 25000-3500t, where t is the number of years since she bought the car.

A. Fill in the table.

| **t, years** | 0 | 1 | 3 | 4.5 | 6.25 |
|---|---|---|---|---|---|
| V, value in dollars | | | | | |

B. Find an appropriate viewing window for this problem scenario.

15. The number of chirps a cricket makes per minute is given by N = 4T-160 where T is the temperature in degrees Fahrenheit.
   A. How many chirps per minute does a cricket make when it is 90°F?
   B. If a cricket makes 45 chirps per minute, what temperature is it?
   C. Does the point (60, 80) satisfy the equation? If so, explain what this point represents in this problem situation.
   D. Find the horizontal-intercept. Explain what this point represents in this problem situation.
   E. Find the vertical intercept. Does this point make sense in the context of this problem? Why or why not?
   F. Give a viewing window that would show a complete graph of this equation in context.
   G. Sketch a complete graph of this equation.

16. The temperature in the desert at 6 am was 65°F. The temperature rose 6 degrees every hour until it reached its maximum value at 4 pm.
   A. Complete the table of values for the temperature, T, at h hours after 6 am.

| hours, h | temperature, T |
|---|---|
| 0 | |
| 5 | |
| 8 | |

   B. Find an equation for the temperature, T, in terms of hours, h, since 6am.
   C. When will the temperature be 90°F?
   D. Sketch a graph for this problem situation. Be sure to use an appropriate viewing window.

For problems 17-20, make a table of values with 4 entries, write an equation for the situation using appropriate variables, and make a complete graph. Find the intercepts and interpret them in context or discuss why the intercept is not reasonable.

17. Jennifer chooses a cell phone plan which has an activation fee of $35 and a monthly fee of $54.99. She must keep the plan for at least 2 years. Write an expression for the total cost of her phone after m months.

18. A deep-sea diver is taking reading as he rises from a depth of 350 feet. He is rising at a rate of 15 feet per minute. Since he starts at 350 feet below sea level his original depth is -350 feet. Write an expression for the diver's depth m minutes after starting his ascent.

19. Peggy won a 50 pound bag of pinto beans. Her family consumes about 0.75 pounds of pinto beans per week. Write an expression for the amount of pinto beans Peggy has left in terms of the number of weeks since she won the bag.

20. Chris makes $7.50 working at a local bookstore. He can only work between 0 and 30 hours per week. Write an expression for the amount Chris earns when he works h hours in a week.

21. A taxi fare in Philadelphia, PA is $2.70 for the first 1/10 mile and then an additional $0.23 per each 1/10 of a mile.
    A. Write an equation for this problem situation.
    B. If Greg takes a taxi 5 miles, what is his taxi fare?
    C. If Amanda has a taxi fare of $10.98, what far did she travel?
    D. Philadelphia also has a flat rate of $28.50 to travel anywhere within the city center. For which distances is the flat rate a better choice?

22. Barry works for a local furniture store. He makes a flat amount of $250 per week plus 7% of his sales for the week.
    A. Write an equation for Barry's salary in terms of his sales.
    B. If Barry sells $3700 worth of furniture, what is his pay for the week?
    C. If Barry needs to make $600 per week, how much furniture does he need to sell?

23. Sharon is shopping for a new washing machine. She has narrowed her choices to two machines. One machine costs $750 plus $32 per year and the other more energy efficient machine costs $900 plus $18 per year.
    A. Write equations for the total cost of each washing machine in terms of the number of years she owns it.
    B. If she decides to buy the more energy efficient model, how long will it be before she starts saving money?

24. A money market account pays 1.75% annual interest on the amount deposited.
    A. Write an equation for the amount in the account after one year in terms of the amount deposited.
    B. If $7500 is deposited, how much will be in the account at the end of the year?
    C. If a person wanted $1000 in the account at the end of the year, how much would they need to deposit?

25. The boiling point of water changes with altitude. At sea level, water boils at 212°F, and the boiling point decreases by approximately 0.002°F for every 1-foot increase in altitude.
    A. Complete the table of values.

| Altitude, a | -500 | 0 | 1000 | 2000 | 3000 | 4000 | 5000 |
|---|---|---|---|---|---|---|---|
| Boiling Point, B | | | | | | | |

    B. Write an equation for the boiling point, B, in terms of the altitude, a, in feet.
    C. At what altitudes is the boiling point less than 204°F?

## 2.2 Equations of Lines and Linear Regression

A linear equation is a first-degree equation. There are 3 forms of the equation of a line.

**Slope-intercept form:** Slope-intercept form is y = mx + b where m is the slope of the line and the point (0, b) is the y-intercept of the line.

**For example:**

1. y = ½ x + 5 is a line with a slope of ½ and a y-intercept of (0,5).
2. y = -4x is a line with a slope of -4 and a y-intercept of (0,0).

**Point-slope form:** Point-slope form is $y – y_1 = m(x - x_1)$ where m is the slope and $(x_1, y_1)$ is any point on the line.

**For example:**

1. y – 4 = 3(x - 1) is a line with a slope of 3 and a point at (1, 4).
2. y – 6 = $-\frac{2}{3}$(x + 4) is a line with slope $-\frac{2}{3}$ and the point (-4, 6).

**Standard form:** Standard form looks like ax + by = c where all terms with variables are on one side of the equation and the constant is on the other. The slope or points would need to be calculated from this form. This form will sometimes be used in application problems.

**For example:** Taylor goes to Taco Bell and buys tacos for $1.29 each and burritos at $1.79 each. She spends $10.53. Write an equation for the amount spent in terms of the number of tacos and burritos she bought.

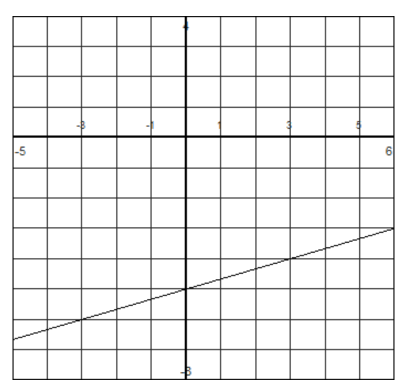
An appropriate equation would be 1.29t + 1.79b = 10.53 where t is the number of tacos and b is the number of burritos she bought.

Both of the first two forms can be used to write the equation of a line given a graph, points, or a table of values. The following examples will find the equation of the lines using first slope-intercept form and then point-slope form.

**Examples:**

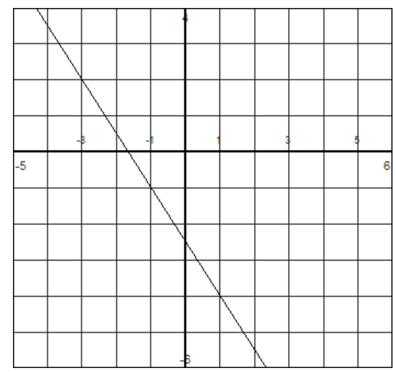
1. Find the equation of the line shown in the graph.

A. Using slope-intercept form: We can see from the graph that the y-intercept is is (0, -5) and the slope is $\frac{1}{3}$. So, the equation of the line is y = $\frac{1}{3}x - 5$.



B. Using point-slope form: The slope is $\frac{1}{3}$. Choose any other point on the graph such as (3, -4) and plug into the equation. y – (-4) = $\frac{1}{3}$(x – 3) or y + 4 = $\frac{1}{3}$ (x – 3). You may choose any point that is on the graph.

2. Find the equation of the line shown in the graph.




A. Slope-intercept form: The slope of the line is $-\frac{3}{2}$ but the y-intercept is not obvious from the graph. In this case, choose any point on the graph and use the equation to find the value of b.

Choosing the point (-3, 2):

y = mx + b Plugging in the value for m and the coordinates of the chosen point gives:

2 = $-\frac{3}{2}(-3) + b$

2 = $\frac{9}{2} + b$

b = $-\frac{5}{2}$

So, the equation of the line is y = $-\frac{3}{2}x - \frac{5}{2}$.

B. Point-slope form: The slope is $-\frac{3}{2}$ and a point on the graph is (-3, 2) so an equation for the line would be y – 2 = $-\frac{3}{2}$(x + 3).

3. The table below shows the height of an object after t minutes have elapsed. Write the equation of the line.

| t, minutes | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| h, height in feet | 120 | 90 | 60 | 30 |

A. Slope-intercept form: The slope is -30 feet per minute. The y-intercept is (0, 120). The equation of the line is h = -30t + 120.

B. Point-slope form: The slope is -30 feet per minute and a point is (0, 120) so an equation would be y – 120 = -30 (t – 0).

4. Three pounds of apples costs $8.37 and 7 pounds of apples costs 19.53. Find the linear equation for the price of apples in terms of the number of pounds of apples purchased.

The slope is $2.79 per pound. Using the point (3, 8.37), an equation would be

C – 8.37 = 2.79(p – 3) which simplifies to C = 2.79p.

**Equations of horizontal and vertical lines:**

A <u>horizontal line</u> has a slope of zero. The equation has the form y = b.

For example, the equation of a horizontal line through the point (0, 4) is y = 4.

A <u>vertical line</u> has an undefined slope. The equation has the form x = a.

For example, the equation of a vertical line through the point (5, -3) is x = 5.

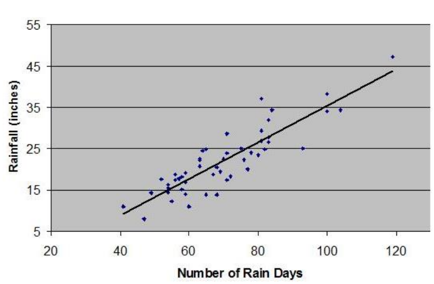
**Linear Regression**

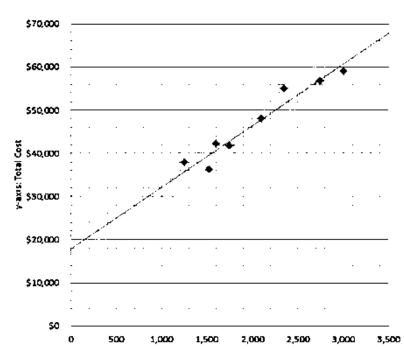
Linear regression is used to predict values from data. Linear regression uses a line of best fit to describe data that is approximately linear. We can approximate a line of best fit using the method described below.

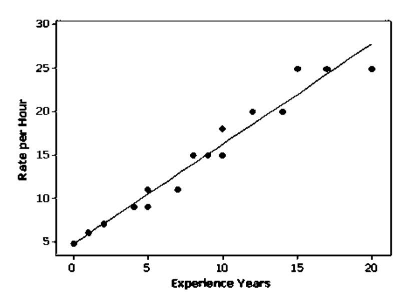
**Steps for linear regression:**

1. Make a graph of the data.
2. Check to see if the data looks linear. It does not need to be in a perfectly straight line but should follow a linear pattern.
3. Using a ruler, draw a line that is as close as possible to all the points on the graph. The line does not need to go through any of the points.
4. Choose two points that are on the regression line using the grid.
5. Write the equation of the line from the chosen points.

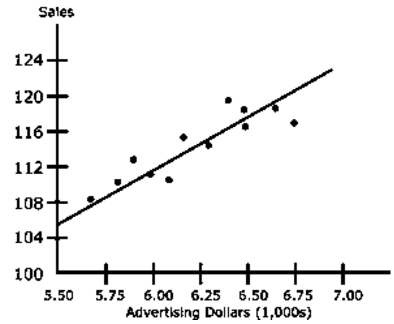
Shown below are some examples of regression lines.



**Example:** The graph shows the sales of a product based on the amount of advertising dollars spent. Find the equation of the regression line shown and use the equation to predict the sales if $7000 is spent on advertising.




The graph appears to have the points (5.5, 106) and (6.5, 116) on the line. To write the equation, we need to find the slope.

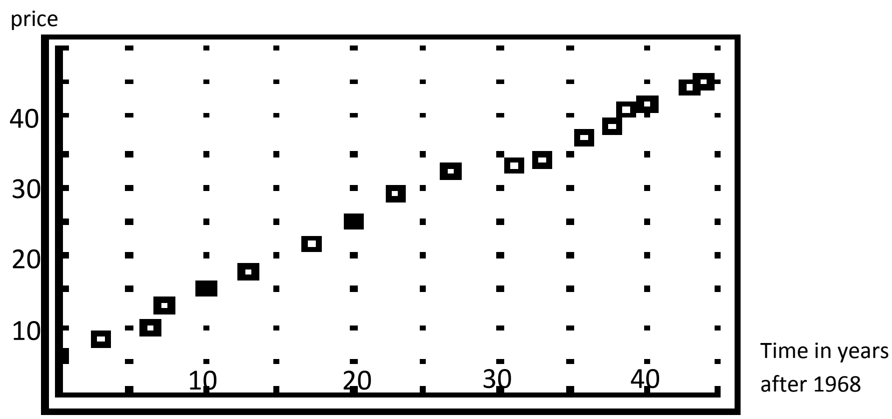
m = $\frac{116-106}{6.5-5.5} = 10$ sales for each $1000 spent on advertising. The equation of the line is s – 106 = 10 (D – 5.5) or s = 10D + 51. The line can be used to predict the sales for $7000 in advertising dollars by letting D = 7. s = 10(7) + 51 = 121 so the sales are predicted to be 121 products.

**Example:**

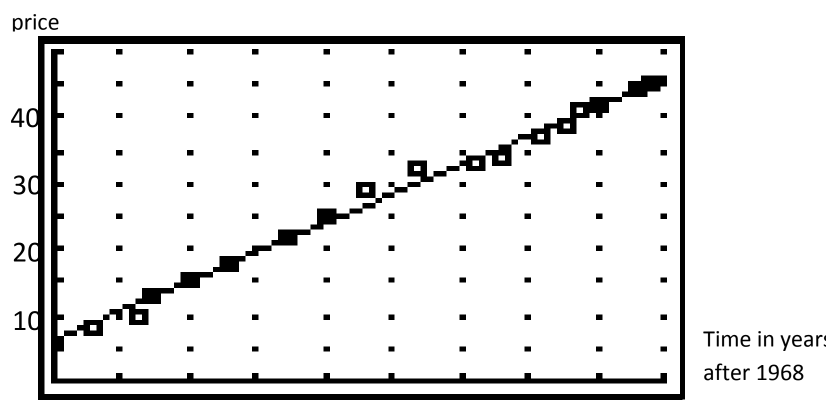
As reported in the Orlando Sentinel, stamp prices for the years 1968 until 2008 are as shown in the table below.

| year | price | year | price | year | price |
|---|---|---|---|---|---|
| 1968 | 6 | 1985 | 22 | 2004 | 37 |
| 1971 | 8 | 1988 | 25 | 2006 | 39 |
| 1974 | 10 | 1991 | 29 | 2007 | 41 |
| 1975 | 13 | 1995 | 32 | 2008 | 42 |
| 1978 | 15 | 1999 | 33 | 2011 | 44 |
| 1981 | 18 | 2001 | 34 | 2012 | 45 |

1. Plot these points. Let t = 0 represent the year 1968.




2. Draw a line of best fit or the regression line.



3. List two points that are on the line you drew. (10, 15) and (15, 20) are on this line.

4. Find the slope of your line and explain what the slope represents in terms of this problem situation. m = $\frac{20-15}{15-10} = \frac{5}{5} = 1$ which means that the price of stamps rise on average 1 cent per year.

5. Find a linear equation for your line.

   P – 15 = 1(t – 10) or P = t + 5 where t is years since 1968

6. Using your equation, predict the price of a stamp in 2013. 2013 is 45 years after 1968 so P = 45 + 5 = 50 or the price of a stamp is predicted to cost $0.50 in 2013.

**Linear Regression on the Graphing Calculator**

According to statistics, the line of best fit is the least-squares regression line which is the line which makes the vertical distances from the data points to the line as small as possible. To find this line by hand would be difficult. The graphing calculator has a built-in function that will find the least-squares regression line. A special note must be made to not just use this function until you have verified that a linear equation will be a good fit for the data.

**Steps to linear regression on the calculator:**

1. Press the **STAT** key. Then **1: Edit**.

2. Type the x-values of your coordinate points into one list and the y-values of your coordinate points into another list. Be sure that the x-coordinate and y-coordinate of a point are in the same row of the lists. The lists should be exactly the same length.

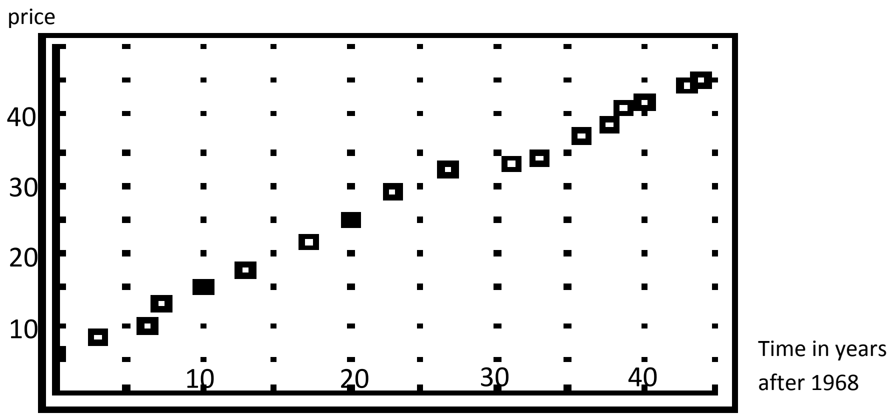
3. To look at the scatterplot:
   Press **2$^{nd}$ Y=** for STAT PLOT
   Choose a **Plot**. Turn it **ON**. The first type is a scatterplot.
   In XLIST: enter the list where you stored the x-values of your coordinate points
   In YLIST: enter the list where you stored the y-values of your coordinate points
   Set a window appropriate to the problem and press **GRAPH**.

4. If a line will provide a good fit for the data, then Press **STAT**. Choose **CALC**. Choose **4: LinReg(ax+b)**. Be sure the appropriate lists for your data are entered. In Store RegEQ: enter **$Y_1$**. Then choose **Calculate**.

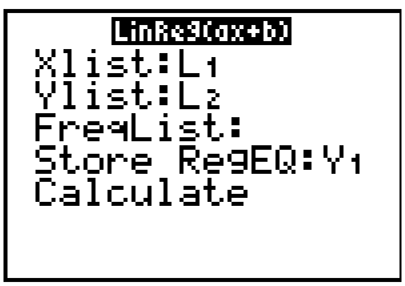
5. You will now have a screen that has **y = ax + b** with the values of **a** and **b** listed, where a is the slope of the regression line and b is the y-intercept of the regression line. You may also have on this screen a value for $r^2$ and r. The letter r represents the correlation coefficient which measures how

good of a fit the least-squares regression line is for the data. Also, the regression equation is now stored in $Y_1$ on your calculator so if you go back to the graph you can see how well the line fits the data on the scatterplot.

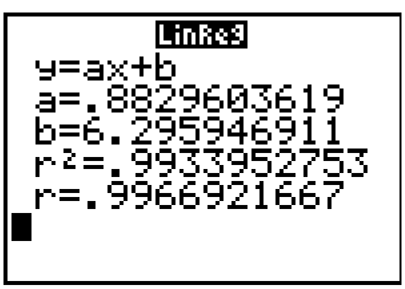
**Returning to the previous example:**
Once you enter the data into the lists, you can make a scatterplot. The scatterplot for the stamp data is shown.




Choose STAT and CALC and 5:LinReg(ax + b). The screen should look like the one shown below if you used lists 1 and 2 for your x and y coordinates.




Choosing Calculate should show the following screen. Your screen may or may not have the values for $r^2$ and r.



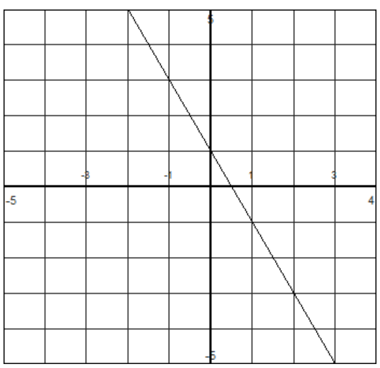

According to the calculator, the least-squares regression line for the stamp data is y = 0.88296x+6.2959. Therefore, the slope of approximately 0.88 tells us that the average rate of change of the price of stamps over those years was 0.88 cents per year. As we set t = 0 to represent the year 1968, the y-intercept of this line means that in 1968 the price of a stamp was approximately 6 cents.

**2.2 Homework:**

Write the equation of the line:

1. With slope of -7 and the point (0, -1)
2. With slope of m = $\frac{2}{7}$ and the point (0, 2)
3. With slope of m =-$\frac{5}{2}$ and the point (3, 1)
4. Through the points (4, 7) and (-1, 5)
5. Through the points (6, -2) and (0, 3)
6. With a slope of 0 and the point (5, 4)
7. Vertical line through the point (3, -6)
8. Horizontal line through the point (1, 5)

9.



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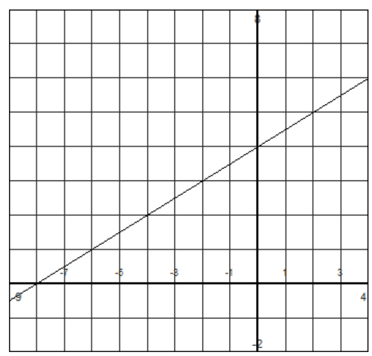

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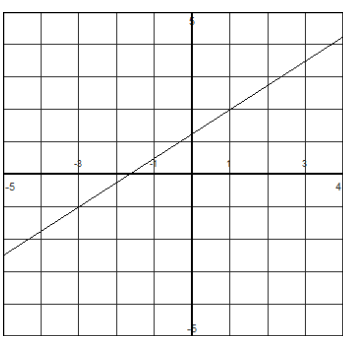

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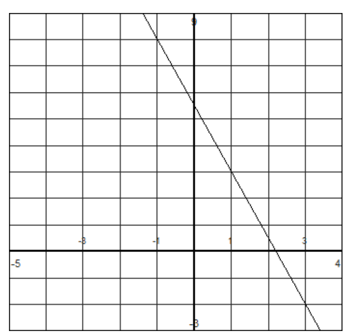
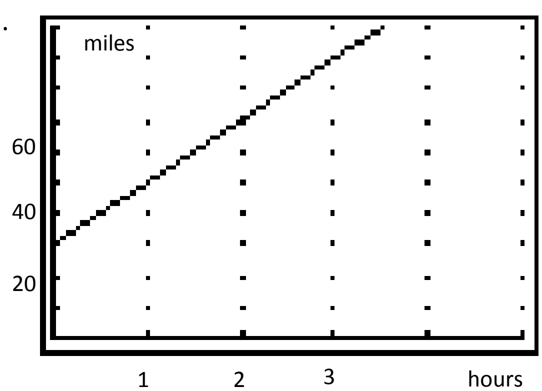
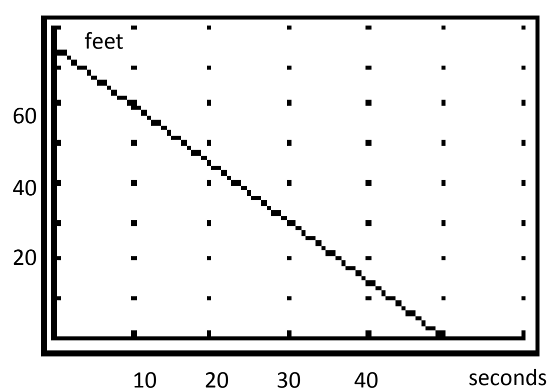


13.




14.



15.

| t, minutes | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| h, height in feet | 120 | 90 | 60 | 30 |

16.

| x, hour | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| y, dollars | 0 | 50 | 100 | 150 |

17.

| x | -3 | 0 | 3 | 6 |
|---|---|---|---|---|
| y | 10 | 12 | 14 | 16 |

18. In 2002, twenty-one percent of married couples in the US lived in single-earner households compared with 63 percent in 1950. Write an equation of a line which finds the percent of single-earner households in terms of the year.

19. A 2-lb box of candy costs \$21 and a 5-lb box of candy costs \$46.50.
    A. Find the slope and interpret.
    B. Find the vertical intercept and interpret.
    C. Write an equation of the line which gives the price of the box of candy in terms of its weight.

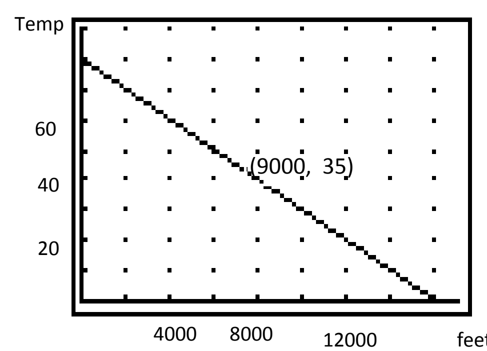
20. The median age of the US population from 1820-1995 was modeled by f(x) = 0.09x-147.1, where x represents the year.
    A. What is the slope of the line and what does it mean?
    B. What is the vertical intercept and what does it mean?

21. The figure below shows the air temperature at certain altitudes. An altitude of 0 feet is sea level. The altitude scale is 2000 feet and the temperature scale is 10 degrees.

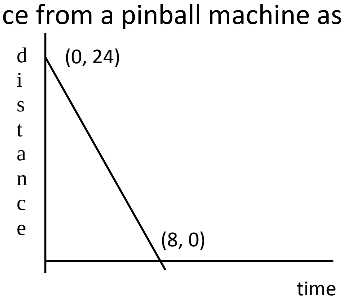
    A. Calculate the slope of the graph. Explain what the slope means in the context of this problem.

    B. Find the vertical intercept. Explain what this point represents in the context of this problem.

    C. Write a linear equation that describes the temperature T, in terms of the altitude, A.




22. Consider the graph below. Time in seconds is graphed on the x-axis and distance in feet is graphed on the y-axis. The graph shows Alisa's distance from a pinball machine as a function of time. Write sentence that describes the graph shown.



23. Consider the following table and choose the sentence that best describes the table.
   A. Kirk walked away from the water ride at a rate of 6 feet per second.
   B. Kirk was 42 feet away from the water ride and walked toward it at a rate of 7 feet per second.
   C. Kirk was 42 feet away from the water ride and walked toward it at a rate of 6 feet per second.
   D. Kirk walked away from the water ride at a rate of 7 feet per second.

| time | Distance from the ride |
|---|---|
| 0 | 42 |
| 1 | 36 |
| 2 | 30 |
| 3 | 24 |
| 4 | 18 |
| 5 | 12 |
| 6 | 6 |
| 7 | 0 |

24. Match each of the following equations to the appropriate sentence.
   A. Kirk walked away from the water ride at a rate of 6 feet per second.
   B. Kirk was 42 feet away from the water ride and walked toward it at a rate of 7 feet per second.
   C. Kirk was 42 feet away from the water ride and walked toward it at a rate of 6 feet per second.
   D. Kirk walked away from the water ride at a rate of 7 feet per second.

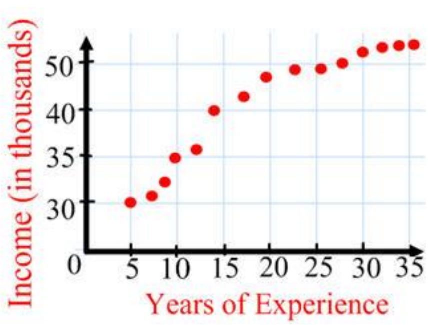
   I) $y = 42 - 6x$   II) $y = 42 - 7x$   III) $y = 6x$   IV) $y = 7x$

25. A newspaper publishing company has fixed costs of $100 for storage and delivery. They have additional costs of $0.50 per paper published.
   A. Fill in the table for costs of publishing the given number of papers.
   B. Write a linear equation that relates the total cost to the number of papers published.
   C. Sketch a graph for this problem situation.
   D. Give an appropriate viewing window for this problem.

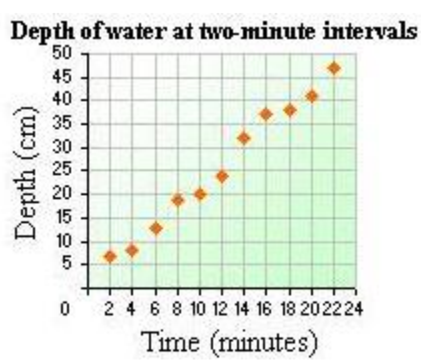
| Number of papers | Total cost |
|---|---|
| 200 | |
| 500 | |
| 1000 | |

For the following, sketch the regression line and find the equation of your line.

26.




27.



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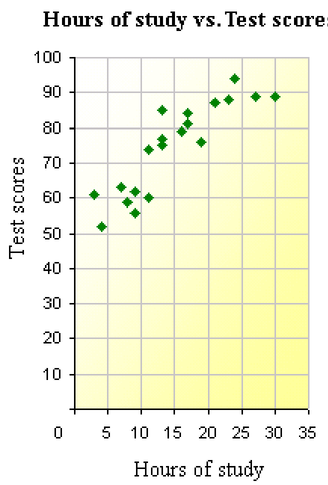
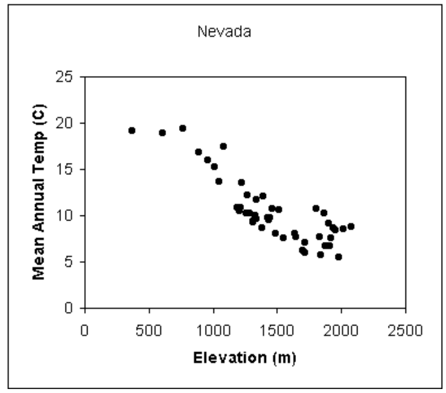



29.




30. The table gives the attendance at Disney World parks in millions as reported in the Orlando Sentinel on August 23, 1998.

| year | Admissions (millions) | year | Admissions (millions) |
|---|---|---|---|
| 1980 | 13.8 | 1989 | 28.2 |
| 1981 | 13.2 | 1990 | 32.8 |
| 1982 | 12.6 | 1991 | 28.2 |
| 1983 | 22.7 | 1992 | 29.6 |
| 1984 | 21.1 | 1993 | 29.1 |
| 1985 | 21.9 | 1994 | 27.6 |
| 1986 | 23.9 | 1995 | 32.8 |
| 1987 | 27.0 | 1996 | 34.4 |
| 1988 | 25.5 | 1997 | 36.3 |

A. Plot these points. **Let t = 0 represent the year 1980.**
B. Use a straight edge to draw a line that "fits" the data.
C. List two points that are on the line you drew.
D. Find the slope of your line. Explain what the slope represents in terms of this problem situation.
E. Find a linear equation for your line.
F. Using your equation, estimate the attendance in the year 2007.
G. Use your calculator to calculate the least-squares regression line.

31. The following data set gives the average heights and weights for American women aged 30-39 (source: The World Almanac and Book of Facts, 1975).

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| inches | 58 | 60 | 62 | 63 | 64 | 65 | 66 | 68 | 70 | 71 | 72 |
| pounds | 115 | 120 | 126 | 129 | 132 | 135 | 139 | 146 | 154 | 159 | 164 |

A. Make a scatterplot of the data.
B. Use your calculator to calculate the least-squares regression line.
C. What is the slope of the regression line? Explain what the slope represents in terms of this problem situation.
D. What is the y-intercept of the regression line? Does this make sense in context?
E. Using the equation, estimate the weight of the average American woman who is 61 inches tall..

For problems 32- 34:

A. Plot the points.
B. If the data appears linear, sketch a regression line. If not, explain.
C. Find the equation of your regression line.
D. Estimate the value in 2013.

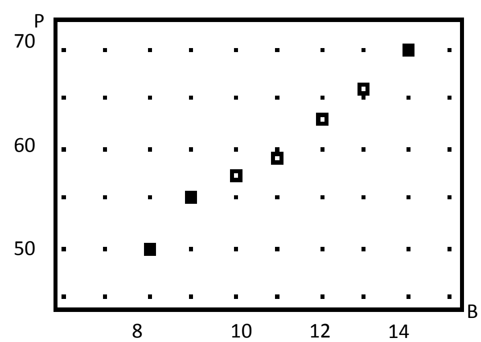
32. A study found that eighty percent of Americans were overweight in 2002, compared with 76 percent in 1998, 69 percent in 1994, 64 percent in 1990, and 59 percent in 1986.

33. The federal minimum wage in 1940 was \$0.25, in 1968 it was \$1.60, in 1980 it was \$3.10, in 1981 it was \$3.35, in 1997 it was \$5.15, and in 2012 it was \$7.25.

34. The number of Bachelor degrees awarded in 1970 was 792,317; in 1980 the number was 929,417; in 1990 the number was 1,051,344; in 2000 the number was 1,244,000; and in 2010 there were 1,399,542 Bachelor degrees awarded.

35. A temperature of 50°F is equal to a temperature of 283°Kelvin. Also, a temperature of 77°F is equal to a temperature of 298°Kelvin. Write a linear equation for the Kelvin temperature, K, in terms of the Fahrenheit temperature, F.

36. The scatterplot shows the approximate relationship between the height of a person, P, and the length of the humerus bone, B (the bone located between the elbow and the shoulder). The scale on the horizontal axis is 1 inch (starting at 6) and the scale on the vertical axis is 5 inches (starting at 45).




A. Use a straight-edge to draw in a line that "fits" the data.
B. Use your line to predict the height of a person whose humerus bone is 7 inches and the height of a person whose humerus bone is 11 inches.
C. Use your answers from part B to find the equation of the line that you drew.

## 2.3 Solving Quadratric Equations

A quadratic equation is any equation which can be written in the form $ax^2 + bx + c = 0$. There are several ways to solve quadratic equations including graphically, numerically, and several algebraic methods. Quadratic equations can have zero, one or two real solutions. If the quadratic equation has zero real solutions, then it will have two complex solutions.

When solving an equation of the type $x^2 = k$, we will get two real solutions of the form $x = \pm\sqrt{k}$. You can see this when solving $x^2 = 25$. Both 5 and -5 when squared will give a result of 25.

**Solving by taking square roots**

In order to solve by taking square roots, the term that is squared must be the only term with a variable and must be able to be isolated on one side of the equal sign.

**Steps:**

1. Isolate the term that is squared on one side of the equal sign.
2. Take the square root of both sides of the equation. Make sure to put a plus or minus.
3. If needed, separate into two equations and solve each.
4. Verify the solutions.

**Examples:**

1. Solve $5x^2 - 7 = 13$ for x.

   **Solution:**

   | | |
   |---|---|
   | $5x^2 - 7 = 13$ | Add 7 to both sides |
   | $5x^2 = 20$ | Divide both sides by 5 |
   | $x^2 = 4$ | Take the square root of both sides (plus or minus) |
   | $\sqrt{x^2} = \pm\sqrt{4}$ | Simplify |
   | $x = \pm 2$ | |

2. Solve $2(3x + 1)^2 = 18$ for x.

   **Solution:**

   | | |
   |---|---|
   | $2(3x + 1)^2 = 18$ | Divide both sides by 2 |
   | $(3x + 1)^2 = 9$ | Take the square root of both sides |
   | $\sqrt{(3x+1)^2} = \pm\sqrt{9}$ | Simplify |
   | $3x + 1 = \pm 3$ | Split into two equations to solve |
   | $3x + 1 = 3$ or $3x + 1 = -3$ | Solve each equation for x |
   | $3x = 2$ or $3x = -4$ | |
   | $x = \frac{2}{3}$ or $x = -\frac{4}{3}$ | |

**Solving by Factoring**

To solve by factoring, the polynomial must be factorable and then we use the zero-factor principle.

**Zero-Factor Principle:** If $a \cdot b = 0$ then $a = 0$ or $b = 0$ or both are equal to zero.

If a polynomial can be factored then this method can be used to find the solutions to the quadratic equation. If the polynomial is not factorable, then you will need to use another method.

**Steps to solve by factoring:**

1. Put the polynomial in standard form (set it equal to zero).
2. Factor the polynomial.
3. Set each factor equal to zero.
4. Solve each equation.

**Examples:**

1. Solve $x^2 + 16x + 48 = 0$ by factoring.

   **Solution:**

   | | |
   |---|---|
   | $x^2 + 16x + 48 = 0$ | Factor the quadratic |
   | $(x + 12)(x + 4) = 0$ | Use the zero-factor principle |
   | $x + 12 = 0$ or $x + 4 = 0$ | Solve each equation |
   | $x = -12$ or $x = -4$ | |

2. Solve $3x^2 – x = 10$ by factoring.

   **Solution:**

   | | |
   |---|---|
   | $3x^2 – x = 10$ | Set the equation equal to zero |
   | $3x^2 – x – 10 = 0$ | Factor the polynomial |
   | $(3x + 5)(x – 2) = 0$ | Set each factor equal to zero |
   | $3x + 5 = 0$ or $x – 2 = 0$ | Solve each equation |
   | $x = -5/3$ or $x = 2$ | |

**Solving by Completing the Square**

The method of completing the square involves rewriting the equation so that the equation can be solved by taking square roots. The equation is first rewritten to contain a perfect square trinomial. Recall that a perfect square trinomial is a trinomial of the form where $x^2 + bx + c = (x + e)^2$ such as $x^2 + 10x + 25 = (x + 5)^2$.

**Steps to complete the square:**

1. Rewrite the equation so that the terms with variables are on one side of the equation and constants are on the other side of the equation.
2. If the leading coefficient is not 1, divide both sides by an appropriate constant so that the leading coefficient is 1.

3. Complete the square by adding an appropriate constant to the side with the variables so that you have a perfect square trinomial. This is done by taking half of the coefficient of x and squaring it.
4. Factor the perfect square trinomial.
5. Solve the equation by taking square roots.

**Examples:**

1. Solve $x^2 - 6x - 4 = 0$ by completing the square.

**Solution:**

| | |
|---|---|
| $x^2 - 6x - 4 = 0$ | |
| $x^2 - 6x = 4$ | First rewrite the equation. |
| $x^2 - 6x + 9 = 4 + 9$ | Since the leading coefficient is 1, complete the square by taking half of the coefficient of x and squaring. $(-6/2)^2 = 9$ Add this to both sides of the equation. |
| $(x - 3)^2 = 13$ | Factor the perfect square trinomial. |
| $x - 3 = \pm\sqrt{13}$ | Take the square root of both sides of the equation. |
| $x = 3 \pm\sqrt{13}$ | Solve for x. |

2. Solve $2x^2 + 8x - 17 = 0$ by completing the square.

**Solution:**

| | |
|---|---|
| $2x^2 + 8x = 17$ | Rewrite the equation. |
| $x^2 + 4x = 17/2$ | Since the leading coefficient is not 1, divide both sides of the equation by the leading coefficient of 2. |
| $x^2 + 4x + 4 = 17/2 + 4$ | Complete the square by taking $(4/2)^2 = 4$ and adding this number to both sides of the equation. |
| $(x + 2)^2 = 25/2$ | Factor the perfect square trinomial. |
| $x + 2 = \pm\sqrt{\frac{25}{2}}$ | Take the square root of both sides of the equation. |
| $x = -2 \pm\sqrt{\frac{25}{2}} = -2 \pm \frac{5\sqrt{2}}{2}$ | Solve for x. Rationalize the denominator for a "simplified" solution. |

3. Solve $3x^2 + 24x + 75 = 0$ by completing the square.

**Solution:**

| | |
|---|---|
| $3x^2 + 24x = -75$ | Rewrite the equation. |
| $x^2 + 8x = -25$ | Divide through by 3 to have a leading coefficient of 1. |
| $x^2 + 8x + 16 = -25 + 16$ | Complete the square by taking $(8/2)^2 = 16$ and adding this number to both sides of the equation. |
| $(x + 4)^2 = -9$ | Factor the perfect square trinomial. |
| $x + 4 = \pm\sqrt{-9}$ | Take the square root of both sides of the equation. Recall that the square root of -1 is the imaginary number *i*. |
| $x = -4 \pm 3i$ | Solve for x. |

**Solving using the Quadratic Formula**

The quadratic formula will solve any quadratic equation. The equation must first be put in standard form $ax^2 + bx + c = 0$.

**Quadratic Formula:** The solutions to the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Derivation of the quadratic formula:** The quadratic equation can be derived by solving the general quadratic $ax^2 + bx + c = 0$ for x using the method of completing the square.

| | |
|---|---|
| $ax^2 + bx = -c$ | Move the constant. |
| $x^2 + \frac{b}{a}x = -\frac{c}{a}$ | Divide both sides of the equation by the leading coefficient. |
| $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$ | Complete the square. $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ |
| $(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$ | Factor the perfect square trinomial on the left and add the terms on the right side of the equation by getting a common denominator. |
| $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$ | Take the square root of both sides of the equation. |
| $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ | Solve for x. |

**Steps to use the quadratic formula:**

1. Put the equation in standard form, or in other words, set the equation equal to zero.
2. Identify the values of a, b, and c.
3. Plug the values of a, b, and c into the quadratic formula.
4. Simplify.
5. Approximate the solutions if necessary.

**Examples:**

1. Solve $20x^2 + 13x + 2 = 0$ using the quadratic formula.

   **Solution:**
   From the equation we can see that a = 20, b = 13, and c = 2.
   Plugging into the quadratic equation, we get:

   $x = \frac{-13\pm\sqrt{13^2-4(20)(2)}}{2(20)}$ Simplifying

   $x = \frac{-13\pm\sqrt{169-160}}{40}$

   $x = \frac{-13\pm\sqrt{9}}{40}$ Taking the square root

$$x = \frac{-13 \pm 3}{40}$$ Splitting into two equations

$$x = \frac{-13+3}{40} = \frac{-10}{40} = \frac{-1}{4} \quad \text{or} \quad x = \frac{-13-3}{40} = \frac{-16}{40} = \frac{-2}{5}$$

The solutions are x =-1/4 and x = -2/5.

2. Solve $2x^2 - 3x + 7 = 0$ using the quadratic formula.

**Solution:**
From the equation we can see that a = 2, b = -3, and c = 7.
Plugging into the quadratic equation, we get:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(7)}}{2(2)}$$ Simplifying

$$x = \frac{3 \pm \sqrt{9-56}}{4}$$

$$x = \frac{3 \pm \sqrt{-47}}{4}$$ Simplifying the square root of -1.

$$x = \frac{3 \pm \sqrt{47}i}{4}$$ The solutions are $x = \frac{3+\sqrt{47}i}{4}$ and $x = \frac{3-\sqrt{47}i}{4}$ .

**Solving Numerically**

To solve a quadratic equation numerically, we use a table. Remember that most quadratic equations have two solutions. To solve $ax^2 + bx + c = 0$ numerically, we let $y_1 = ax^2 + bx + c$ and look for the values of x which have an output value of zero.

**Example:**

Solve $2x^2 – 5x + 2 = 0$ numerically.
**Solution:**
Set $y_1 = 2x^2 – 5x + 2$ and make a table. Find the values where $y_1 = 0$.

| X | Y1 | |
|---|---|---|
| 0 | 2 | |
| .5 | 0 | |
| 1 | -1 | |
| 1.5 | -1 | |
| 2 | 0 | |
| 2.5 | 2 | |
| 3 | 5 | |

Press + for ΔTbl

The solutions are x = 0.5 and x = 2.

Solving quadratic equations numerically is often difficult because the solutions are not usually integer values. A better way to use technology to solve quadratic equations is to solve them graphically.

**Solving Graphically**

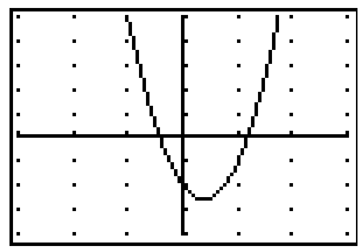
When solving $ax^2 + bx + c = 0$ graphically, we first set $y_1 = ax^2 + bx + c$. Notice that the solutions will be at the values for which $y_1 = 0$. Points which have a y-value of zero are the x-intercepts of a graph. So, to find the solutions we simply need to find the x-intercepts of the graph. This can be done on the calculator in two ways. The first is to set $y_2 = 0$ and find the intersection of the two graphs. The second way is to use the **ZERO** key which is located in the **CALC** menu (**2$^{nd}$ TRACE**). If the graph is already given, then estimate the x-intercepts from the graph.

**Examples:**

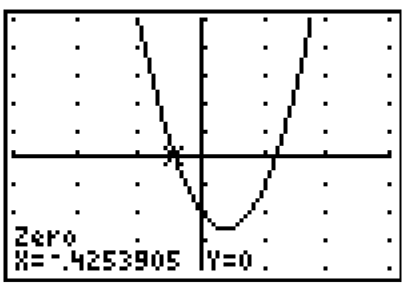
1. Solve $4x^2 - 3x - 2 = 0$ graphically.

   **Solution:**

   Set $Y_1 = 4x^2 - 3x - 2$ and find an appropriate viewing window to see the x-intercepts.

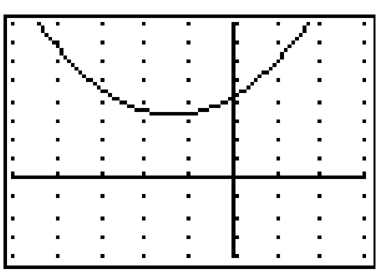
   Use the **ZERO** command twice to find the two x-intercepts.


   So, the solutions are x = -0.4254 and x = 1.1754.

2. Solve $0.5x^2 + 1.35x + 4.2 = 0$ graphically.

   **Solution:**
   Set $y_1 = 0.5x^2 + 1.35x + 4.2$ and find an appropriate viewing window.

   As seen on the graph, this graph has no x-intercepts. Therefore, the equation

$0.5x^2 + 1.35x + 4.2 = 0$ has no real solutions. It will have complex solutions. If you wanted to find the complex solutions, we would need to use the quadratic formula or complete the square.

**Equations Reducible to Quadratic Form**

Some equations can be solved with the techniques of this section even though the equation is not quadratic. This process involves making a substitution so that the new equation is quadratic. Such equations that can be rewritten are said to be quadratic in form. Be sure to solve the original equation once you have completed solving the quadratic equation. The substitution usually involves the variable u.

**Examples:**

1. Solve $x^4 - 3x^2 - 10 = 0$.

   Looking at this equation, if we substitute $u = x^2$, the equation becomes $u^2 - 3u - 10 = 0$. This is a quadratic equation which can be solved by factoring.

   $u^2 - 3u - 10 = 0$

   $(u - 5)(u + 2) = 0$

   $u = 5$ or $u = -2$

   Now, remember that $u = x^2$. This gives $x^2 = 5$ and $x^2 = -2$. Solving for x, we find that $x = \pm\sqrt{5}$ or $x = \pm\sqrt{2}i$.

2. Solve $x + 4\sqrt{x} - 1 = 0$.

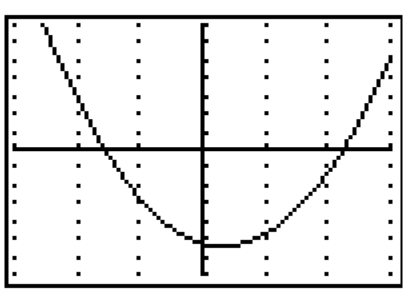
   If we choose $u = \sqrt{x}$, then this equation becomes $u^2 + 4u - 1 = 0$. This is a quadratic equation that can be solved with the quadratic formula.

   $u = \frac{-4\pm\sqrt{4^2-4(1)(-1)}}{2(1)} = \frac{-4\pm\sqrt{20}}{2} = -2 \pm \sqrt{5}$

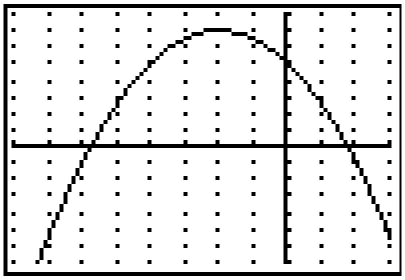
   Now, remember that $u = \sqrt{x}$, so $\sqrt{x} = -2 \pm \sqrt{5}$ or $x = (-2 \pm \sqrt{5})^2$.

**2.3 Homework:**

1. Estimate the solutions to $1.5x^2 - x - 5.25 = 0$ using the graph of $y = 1.5x^2 - x - 5.25$ shown.



2. Estimate the solutions to $-0.5x^2 - 2x + 5 = 0$ using the graph of $y = -0.5x^2 - 2x + 5$ shown.



3. Solve graphically using your graphing calculator $-3x^2 + 4x + 10 = 0$.

4. Solve graphically using your graphing calculator $0.2x^2 - 5x + 10 = 0$.

5. Solve graphically using your graphing calculator $1.2x^2 + 4x - 7.5 = 0$.

6. Solve graphically using your graphing calculator $-2x^2 - x + 8 = 0$.

7. Solve $x^2 + 9x + 20 = 0$ numerically given the table of $y_1 = x^2 + 9x + 20$ as shown.

| X | $Y_1$ |
|---|---|
| -6 | 2 |
| -5 | 0 |
| -4 | 0 |
| -3 | 2 |
| -2 | 6 |
| -1 | 12 |
| 0 | 20 |

Press + for ΔTbl

8. Solve $x^2 + x - 2 = 0$ numerically given the table of $y_1 = x^2 + x - 2$ as shown.

| X | $Y_1$ |
|---|---|
| -2 | 0 |
| -1 | -2 |
| 0 | -2 |
| 1 | 0 |
| 2 | 4 |
| 3 | 10 |
| 4 | 18 |

X=4

9. Solve $x^2 + 4x - 12 = 0$ numerically. Show a table.

10. Solve $2x^2 - 18 = 0$ numerically. Show a table.

Solve the following by taking square roots.

11. $6x^2 + 7 = 19$

12. $4x^2 - 3 = 11$

13. $5(x + 1)^2 - 2 = 18$

14. $2(3x - 1)^2 + 1 = 21$

Solve the following by factoring.

15. $6x^2 - 13x - 5 = 0$

16. $3x^2 - 14x - 5 = 0$

17. $x^2 - 12x + 35 = 0$

18. $6x^2 + 46x + 28 = 0$

Solve by completing the square.

19. $x^2 - 16x - 40 = 0$

20. $x^2 + 5x + 15 = 0$

21. $4x^2 + 24x + 13 = 0$

22. $6x^2 - 13x - 5 = 0$

Solve the following by the quadratic formula.

23. $4x^2 - 2x - 3 = 0$

24. $2x^2 + 5x + 7 = 0$

25. $-3x^2 - 7x + 9 = 0$

26. $x^2 + 9x - 11 = 0$

Solve by any method.

27. $7(2x + 1)^2 - 9 = 5$

28. $3x^2 - 17 = 0$

29. $x^2 - 8x + 7 = 0$

30. $3x^2 - 3x + 2 = 0$

31. $4x^2 - 9 = 0$

32. $5x^2 - 9x + 3 = 0$

Solve the following.

33. $2x^4 - 9x^2 + 4 = 0$

34. $3x^4 + x^2 - 5 = 0$

35. $2x - 7\sqrt{x} + 3 = 0$

36. $x - 5\sqrt{x} - 2 = 0$

**2.4 Graphing Quadratic Functions**

In the previous section, we saw that there are several ways to solve a quadratic equation of the form $ax^2 + bx + c = 0$. We also saw when solving these equations graphically, that the solutions were the x-intercepts of the graph. We can use these ideas to help graph equations of quadratic functions and also to write the equations of quadratic functions.

Recall when sketching complete graphs, the graphs should include the intercepts if the intercepts make sense in context of the problem. We can find the intercepts of quadratic graphs using the rules we have used for finding horizontal and vertical intercepts in previous sections. Given the equation $f(x) = x^2 + 5x + 4$, the vertical intercept is found by setting x =0 in the equation. This gives a vertical intercept of (0, 4). The horizontal intercepts are found by setting f(x) = 0 in the equation. This gives $x^2 + 5x + 4 = 0$ which can be solved by factoring (or the quadratic formula). Solving gives x = -4 or x = -1. So the horizontal intercepts are (-4, 0) and (-1, 0).

**Examples:**

Find the horizontal and vertical intercepts of:

1. $f(x) = 6x^2 + 15x + 6$
2. $g(x) = 2x^2 – 5x – 12$
3. $h(x) = x^2 + 8$

**Solutions:**

1. To find the vertical intercept, set x = 0:

   $f(0) = 6(0)^2 + 15(0) + 6 = 6$ so the vertical intercept is (0, 6).

   To find the horizontal intercepts, set the equation equal to 0.

   $6x^2 + 15x + 6 = 0$

   $3(2x^2 + 5x + 2) = 0$

   $3(2x + 1)(x + 2) = 0$

   $2x + 1 = 0$ or $x + 2 = 0$

   $x = -½$ or $x = -2$

   The horizontal intercepts are (- ½ , 0) and (-2, 0).

2. To find the vertical intercept:

   $g(0) = 2(0)^2 – 5(0) – 12 = -12$ so the vertical intercept is (0, -12).

   To find the horizontal intercepts, set the equation equal to 0.

   $2x^2 – 5x – 12 = 0$

   $(2x + 3)(x – 4) = 0$

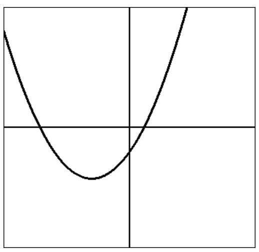
   $2x + 3 = 0$ or $x – 4 = 0$

   $x = -3/2$ or $x = 4$

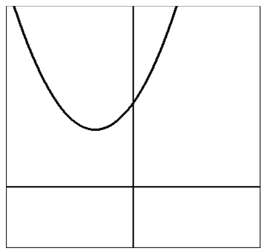
   The horizontal intercepts are (-3/2, 0) and (4, 0).

3. To find the vertical intercept, set x = 0.
   $h(0) = 0^2 + 8 = 8$ so the vertical intercept is (0, 8).
   To find the horizontal intercepts, set the equation equal to 0.
   $x^2 + 8 = 0$
   $x^2 = -8$
   $x = \pm\sqrt{-8}$ This equation has no real solutions.
   Therefore, the graph has no horizontal intercepts.

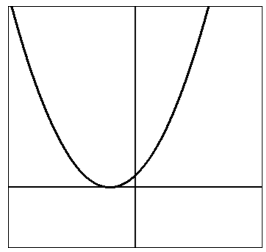
All quadratic equations will have a vertical intercept. They may have either 0, 1, or 2 horizontal intercepts as illustrated in the graphs shown below.



2 x-intercepts

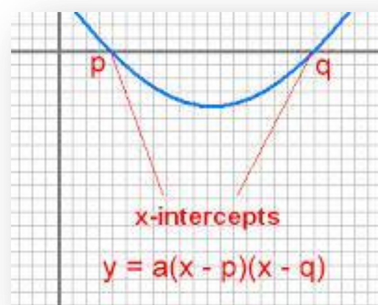


0 x-intercepts



1 x-intercept

**Graphing Quadratic Functions in Intercept Form**




In general, the **intercept form of a quadratic equation** is:

$y = a(x-p)(x-q)$

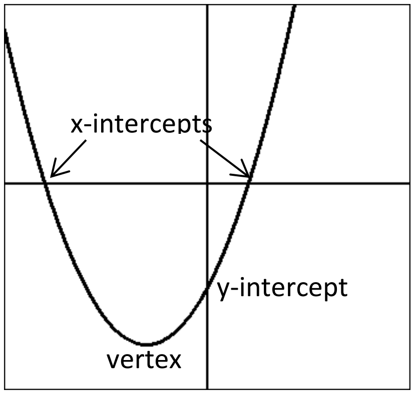
where (p, 0) and (q, 0) are the x-intercepts of the graph. The value of a determines the width of the parabola and its concavity. The intercept form of a quadratic equation is the form of a quadratic equation by which you can easily tell the x intercepts of the quadratic equation.

**Example:**

The x-intercepts of parabolas can be obtained from their quadratic equations in intercept forms as follows:

| **Quadratic Equation in intercept form** | **x-intercepts of its parabola** |
|---|---|
| y = (x - 3)(x - 4) | (3, 0) and (4, 0) |
| y = 3x(x - 5) | (0, 0) and (5, 0) |
| y = (x + 3)(x + 5) | (-3, 0) and (-5, 0) |

To graph a complete graph of a quadratic function, we should show the intercepts and the vertex of the parabola.



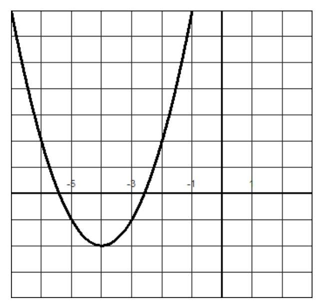

**Graphing Quadratic Functions in Vertex Form**

Vertex form is $f(x) = a(x – h)^2 + k$. Recall that the graph of $f(x) = a(x – h)^2 + k$ is a transformation of the graph of $f(x) = x^2$. The value of **a** determines the width and concavity of the graph. The value of **h** gives the horizontal shift and the value of **k** gives the vertical shift of the graph.

**Examples:**

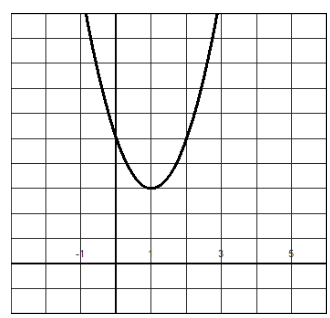
1. Graph $f(x) = (x + 4)^2 – 2$ and identify the vertex.

   **Solution:**
   This graph will be shifted 4 units to the left and 2 units downward. The vertex will be (-4, -2).



2. Graph $g(x) = 2(x – 1)^2 + 3$ and identify the vertex of the graph.

   **Solution:**
   This graph will be shifted 1 unit to the right and 3 units upward. The graph is stretched vertically by 2. The vertex is (1, 3).



**The vertex of a quadratic function in vertex form $f(x) = a(x – h)^2 + k$ is at the point (h, k).**

For example, the vertex of $f(x) = -(x + 2)^2 + 5$ is at (-2, 5) and the vertex of $g(x) = 4(x – 7)^2 - 1$ is at the point (7, -1).

Recall from the discussion on transformations that the value of **a** affects the points on either side of the vertex.

We can now find all important points of the graph of a parabola when the equation is in vertex form.

**Example:**

Find the vertex and intercepts of $h(x) = 3(x + 1)^2 – 6$.

The y-intercept is found by setting $x = 0$. $h(0) = 3(0 + 1)^2 – 6 = 3 – 6 = -3$ so the y-intercept is (0, -3).

The x-intercepts are found by setting $h(x) = 0$.

$3(x + 1)^2 – 6 = 0$

$3(x + 1)^2 = 6$

$(x + 1)^2 = 2$

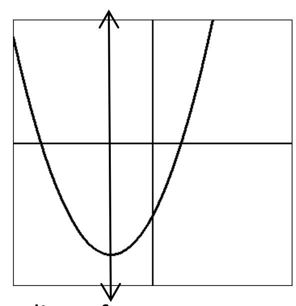
$x + 1 = \pm\sqrt{2}$

$x = -1 \pm \sqrt{2}$

So the x-intercepts are $(-1 + \sqrt{2}, 0)$ and $(-1 - \sqrt{2}, 0)$.

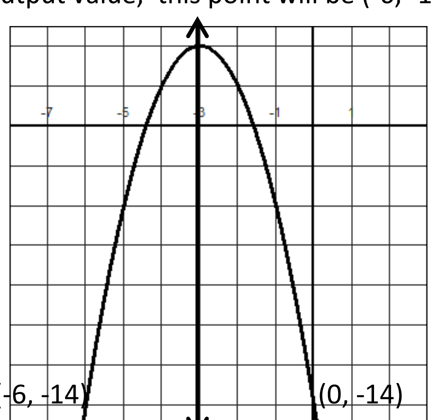
The vertex is (-1, -6). Note, the graph is stretched vertically by 3.

When sketching a parabola, we can use symmetry to find other points on the graph. Recall that the graph of any parabola is symmetric about a vertical line that goes through its vertex. That line is called the **line of symmetry.** If you find a point on one side of the line of symmetry, the graph must have a point equidistant on the other side of the line which has the same output value.



line of symmetry

For example, $f(x) = -2(x + 3)^2 + 4$ has a vertex at (-3, 4). The line of symmetry for this graph is $x = -3$. We can also find the y-intercept which is (0, -14). This point is three units to the right of the line of symmetry. A point symmetric to this one will be three units to the left of the line of symmetry and have the same output value; this point will be (-6, -14).




line of symmetry

x = -3

**Graphing Quadratic Functions in Standard form**

A quadratic function in standard form is $f(x) = ax^2 + bx + c$. First, let's investigate how the values of **a**, **b**, and **c** affect the graph of the parabola.

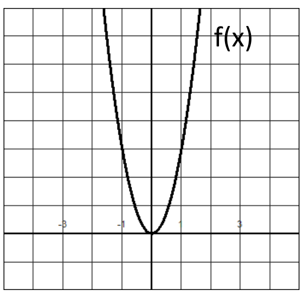
Previously, when discussing transformations, we found that the value of **a** affects the width of the parabola and its concavity. If $|a| > 1$, then the graph is stretched vertically. If $0 < |a| < 1$, then the graph is shrunk vertically. If $a > 0$, then the graph is concave up (opens upward) and if $a < 0$, then the graph is concave down (opens downward).

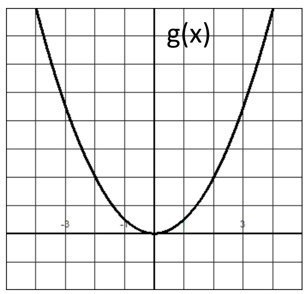
**Example:**

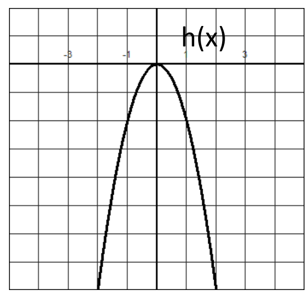
Given the equations $f(x) = 3x^2$ and $g(x) = ½\, x^2$, and $h(x) = -2x^2$, describe the graphs and sketch.

**Solution:**

$f(x) = 3x^2$ is stretched vertically by 3 and is concave up
$g(x) = ½\, x^2$ is shrunk vertically by ½ and is concave up
$h(x) = -2x^2$ is stretched vertically by 2 and is concave down



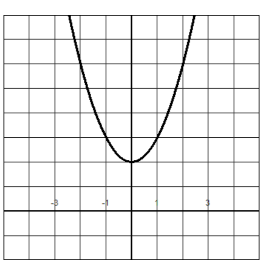

The value of **c** determines the y-intercept of the graph. To find a y-intercept, we set $x = 0$. So, $f(0) = a(0)^2 + b(0) + c = c$. The point $(0, c)$ is the y-intercept of the graph. For example, $f(x) = 4x^2 – 6x + 1$ has a y-intercept at $(0, 1)$ and $g(x) = 2x^2 – 5$ has a y-intercept at $(0, -5)$.

The coefficient **b** affects the vertex of the parabola. If $b = 0$, then the vertex of the parabola will be on the y-axis. If $b \neq 0$, then the vertex is shifted off of the y-axis.

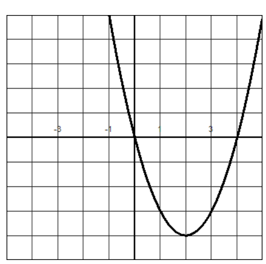
**Example:**

Sketch the following graphs and identify the vertex.
$f(x) = x^2 + 2$, $g(x) = x^2 – 4x$, $h(x) = x^2 + 8x – 1$

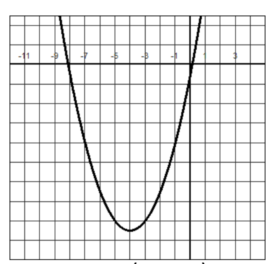
**Solution:**



vertex: (0, 2)



vertex: (2, -4)



vertex: (-4, -17)

Notice in the above example that the first graph has a vertex on the y-axis at (0, 2) and the other two graphs have a vertex that is not on the y-axis. The vertex on these graphs has not just been shifted right or left though. We will find a formula for the vertex later in this section.

Therefore, given an equation in standard form we should be able to identify the concavity, the width, the y-intercept and whether or not the vertex is shifted off the y-axis.

**Examples:**

Given the following equations, describe what you know about the graphs based on the values of **a**, **b**, and **c**.

1. $f(x) = 2x^2 - 3x + 7$
2. $g(x) = -\frac{1}{2}x^2 - 5$
3. $h(x) = \frac{2}{3}x^2 + 2x$

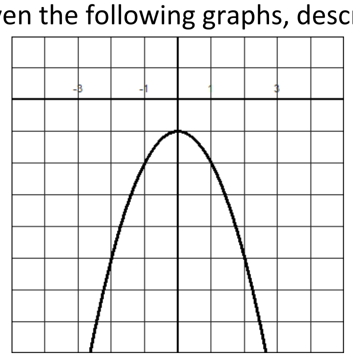
**Solution:**

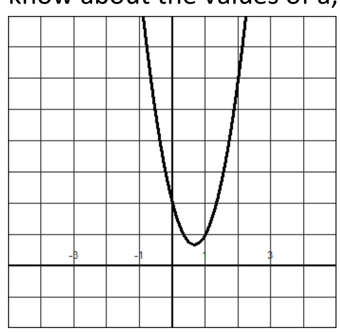
1. f(x) is concave up (a > 0) and stretched ($|a| > 1$). The y-intercept is at (0, 7) and the vertex will not be on the y-axis ($b \neq 0$).
2. g(x) is concave down (a < 0) and shrunk ($|a| < 1$). The y-intercept is at (0, -5) and the vertex will be on the y-axis (b =0).
3. h(x) is concave up (a > 0) and shrunk ($|a| < 1$). The y-intercept is at (0, 0) and the vertex will not be on the y-axis ($b \neq 0$).

Given a graph, we should be able to describe the appropriate values for **a**, **b**, and **c**.

**Examples:**

Given the following graphs, describe what you know about the values of a, b, and c.

1. 

2. 

**Solutions:**

1. Since the graph is concave down, a < 0. The vertex is on the y-axis so b = 0. The y-intercept is below the x-axis so c < 0.
2. The graph is concave up so a > 0. The vertex is not on the y-axis so $b \neq 0$. The y-intercept is above the x-axis so c > 0.

To find the vertex of a parabola when the equation is in standard form, we use the formula x = - $\frac{b}{2a}$ to find the x-coordinate of the vertex. Then, we use the equation to find the value of f(x).

**Examples:**

Find the vertex of the following functions.

1. $f(x) = 2x^2 + 12x - 5$
2. $g(x) = -x^2 + 5x + 2$

**Solutions:**

1. The x-value of the vertex is at x = - $\frac{b}{2a}$. For this equation, a = 2 and b = 12 so the x-value of the vertex is x = $-\frac{12}{2(2)} = -3$.

   Use the equation to find the y-value of the vertex. $f(-3) = 2(-3)^2 + 12(-3) - 5 = 18 - 36 - 5 = -23$.

   The vertex is (-3, -23).

2. For this equation, a = -1 and b = 5 so the x-value of the vertex is x = $-\frac{5}{2(-1)} = \frac{5}{2}$.

   Use the equation to find the y-value of the vertex. $g(\frac{5}{2}) = -(\frac{5}{2})^2 + 5(\frac{5}{2}) + 2 = \frac{33}{4}$.

   The vertex is $(\frac{5}{2}, \frac{33}{4})$.

Previously, we found the intercepts of parabolic functions written in standard form. Using the vertex and the intercepts, we can sketch a complete graph.

**Example:**

Sketch a complete graph of $f(x) = 3x^2 - 6x - 2$.

**Solution:**

The y-intercept is (0, -2). We also know the graph should be concave up and stretched by 3.

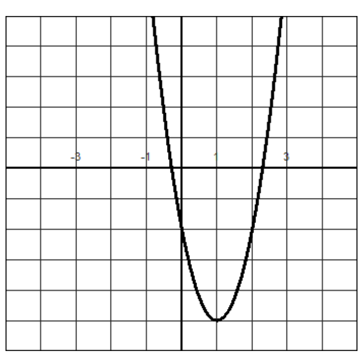
The x-intercepts can be found using the quadratic formula.

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-2)}}{2(3)} = \frac{6 \pm \sqrt{60}}{6}$ so the x-intercepts are (-0.29, 0) and (2.29, 0).

The vertex is:

$x = -\frac{-6}{2(3)} = 1$, $f(1) = 3(1)^2 - 6(1) - 2 = -5$ so the vertex is (1, -5).

The graph is:



**Rewriting Equations from Standard Form to Vertex Form**

To change an equation from standard form to vertex form, we use a technique called completing the square.

**Steps to complete the square:**

1. Put the terms in descending order. Group the first two terms.
2. If the leading coefficient is not 1, factor the leading coefficient from the first two terms.
3. Using $x^2 + bx$, complete the square by taking half of b and squaring the result. Add and subtract this value. If the leading coefficient is not 1, you will need to multiply one of these by the leading coefficient to separate it from the perfect square trinomial.
4. Write $x^2 + bx + c$ as $(x + b/2)^2$.
5. Collect like terms.

**Examples:**

1. Rewrite $f(x) = x^2 + 8x – 1$ in vertex form and identify the vertex.

| | |
|---|---|
| $f(x) = (x^2 + 8x) – 1$ | Group the first two terms. Leading coefficient is 1. |
| $f(x) = (x^2 + 8x + 16) – 16 – 1$ | Since b = 8, then $(8/2)^2 = 16$ so we add and subtract 16. |
| $f(x) = (x + 4)^2 – 17$ | Factor $x^2 + 8x + 16$ into $(x + 4)^2$. |

The equation is now in vertex form. The parabola has a vertex of (-4, -17).

2. Rewrite $f(x) = 2x^2 – 10x + 8$ in vertex form and identify the vertex.

| | |
|---|---|
| $f(x) = (2x^2 – 10x) + 8$ | Group the first two terms. The leading coefficient is 2, so factor. |
| $f(x) = 2(x^2 – 5x) + 8$ | Factor the 2 out of the first two terms. |
| $f(x) = 2(x^2 – 5x +25/4 – 25/4) + 8$ | Since b = 5, then $(5/2)^2 = 25/4$, so add and subtract 25/4. |
| $f(x) = 2(x – 5/2)^2 – 25/2 + 8$ | Factor $x^2 – 5x + 25/4$ into $(x – 5/2)^2$. Multiply -25/4 by the coefficient of 2. |
| $f(x) = 2(x – 5/2)^2 – 9/2$ | Collect like terms of -25/2 and 8. |

The equation is now in vertex form. The parabola has a vertex of (5/2, -9/2).

## 2.4 Homework:

Find the vertical and horizontal intercepts of the following if they exist.

1. $f(x) = x^2 - 25$
2. $F(x) = x^2 + 11x + 28$
3. $g(x) = 2x^2 + x - 3$
4. $Y = 2x^2 - 7x + 5$
5. $h(x) = 12x^2 - 32x + 20$
6. $y = 2(x + 6)^2 - 32$
7. $f(x) = x^2 + 3$
8. $F(x) = 2x^2 - 7x + 1$
9. $h(x) = -x^2 + 3x - 11$
10. $y = 4(x - 2)^2 + 6$

Given the equations, identify the vertex and the line of symmetry.

11. $f(x) = (x + 7)^2 - 2$
12. $g(x) = (x + 5)^2 + 3$
13. $f(x) = 2(x - 3)^2 - 1$
14. $h(x) = -(x + 1)^2 - 9$

Given the equations, identify all intercepts and the vertex of the graph.

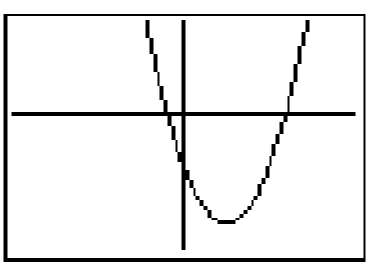
15. $f(x) = -2(x + 6)^2 - 2$
16. $h(x) = 3(x + 2)^2 - 9$
17. $g(x) = 1/3\ (x - 3)^2 - 4$
18. $f(x) = -(x + 5)^2 - 8$

Given the equations, sketch the graph and identify the vertex and the intercepts of the graph.

19. $f(x) = (x + 5)^2 + 2$
20. $g(x) = 2(x - 7)^2 - 8$
21. $f(x) =- ½\ (x - 1)^2 - 5$
22. $f(x) = -(x + 3)^2 - 9$

For the following graphs, identify whether the values of a and c are positive, negative, or zero. Identify whether b is equal to zero or not equal to zero.

23.



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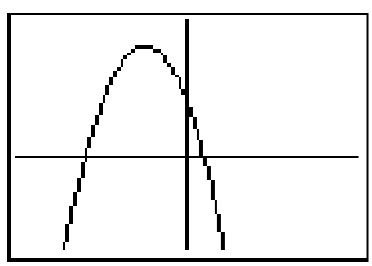

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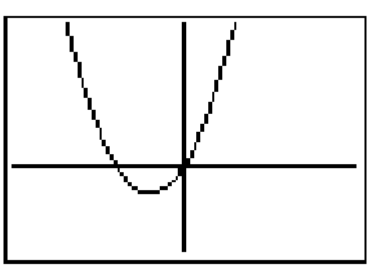
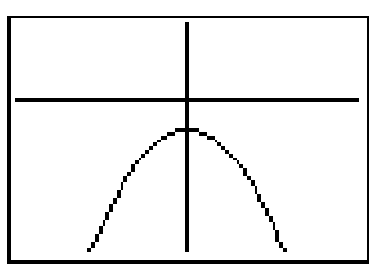


26.



Given the following equations, describe what you know about the graphs based on the values of a, b, and c.

27. $f(x) = -4x^2 - 3x + 11$

28. $g(x) = 6x^2 + 1$

29. $h(x) = \frac{2}{3}x^2 + 2x - 5$

30. $h(x) = -\frac{1}{2}x^2 + x + 3$

Given the following equations, identify the vertex and the intercepts and sketch the graph.

31. $f(x) = 4x^2 - 5$

32. $g(x) = 6x^2 + x - 5$

33. $h(x) = \frac{1}{3}x^2 + 2x$

34. $f(x) = -x^2 + 4x + 3$

35. Given $y = -x^2 + 3x + 4$, find:

A. the y-intercept.
B. the x-intercepts.
C. the vertex.
D. Is the graph concave up or down?
E. Is the graph stretched, shrunk, or standard width?
F. Sketch the graph.

36. Given $y = 2x^2 - 8x + 4$, find:

A. the y-intercept.
B. the x-intercepts.
C. the vertex.
D. Is the graph concave up or down?
E. Is the graph stretched, shrunk, or standard width?
F. Sketch the graph.

37. Given $y = \frac{1}{2}x^2 + 3x - 1$, find:

A. the y-intercept.
B. the x-intercepts.
C. the vertex.
D. Is the graph concave up or down?
E. Is the graph stretched, shrunk, or standard width?
F. Sketch the graph.

Rewrite the following in vertex form.

38. $f(x) = x^2 + 12x + 5$

39. $f(x) = x^2 + 7x - 10$

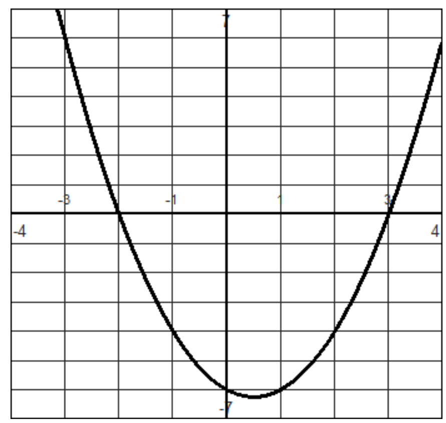
40. $f(x) = 3x^2 - 12x + 17$

41. $f(x) = 2x^2 + 20x - 11$

## 2.5 Writing Equations of Quadratic Functions

Quadratic equations can be written in three ways. One way is to use the vertex and one point on the graph and write the equation using the vertex form of the equation. Another way is to use the x-intercepts and one other point on the graph and write the equation in standard form. The third way is to use any three points on the graph and use the elimination method to solve a system of equations to find the equation in standard form.

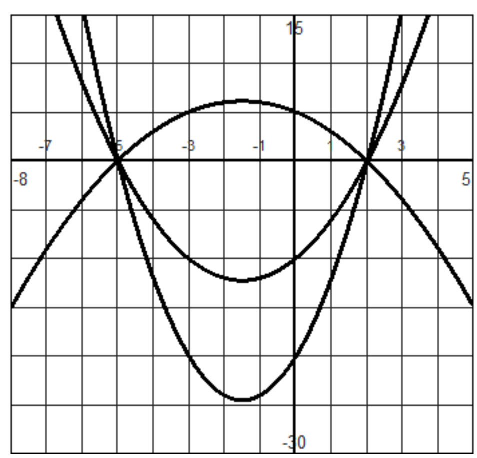
We can write an equation of a quadratic function easily if we can identify the x-intercepts of the graph. This equation can be derived by reversing the steps we used to solve by factoring. Look at the graph shown to the right.




Let us assume that the graph has not been stretched or shrunk. Therefore, we will assume a = 1. This graph has x-intercepts at (-2, 0) and (3, 0). Therefore, when solving by factoring we would have gotten solutions of
x = -2 and x = 3.
Rewriting these equations so they are set equal to zero gives (x + 2) =0 and (x – 3) = 0. This means the quadratic equation in factored form was (x + 2)(x – 3) = 0.
If you would like the equation in standard form, multiply to get $x^2 - x - 6 = 0$. Therefore, the equation of the graph shown is $f(x) = x^2 - x - 6$.

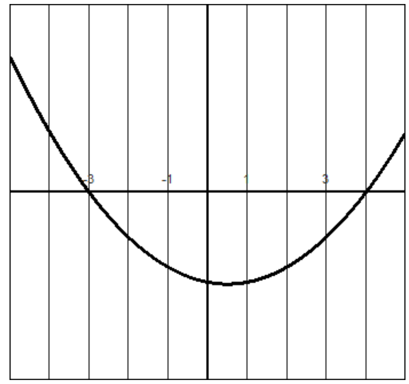
Sometimes there is more to the process of writing the equation than identifying the x-intercepts, let's look at the graphs of:

1. $y = x^2 + 3x - 10 = (x + 5)(x - 2)$
2. $y = 2(x^2 + 3x - 10) = 2(x + 5)(x - 2)$
3. $y = -\frac{1}{2}(x^2 + 3x - 10) = -\frac{1}{2}(x + 5)(x - 2)$




Notice that all of the graphs shown above have the same x-intercepts at (-5, 0) and (2, 0). The graphs vary in width and concavity. To write an equation of a graph, we need to look at more than the x-intercepts of the graph because the graph may have been stretched, shrunk, and/or reflected.

For example, given the graph shown at the right, we can see the x-intercepts are (-3, 0) and (4, 0). To write an equation for the function, we reverse the steps of solving using the zero-factor principle.

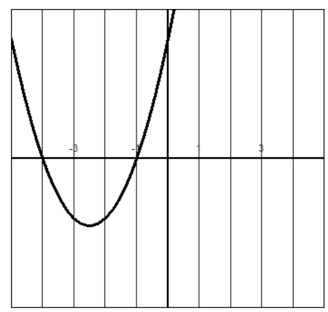


$x = -3$ and $x = 4$
$x + 3 = 0$ and $x - 4 = 0$
$(x + 3)(x - 4) = 0$
$x^2 - x - 12 = 0$

The equation of the graph shown could be $y = x^2 - x - 12$. But it could also be $y = a(x^2 - x - 12)$ where **a** is any positive number. We know **a** has to be a positive number because the graph is concave up but we do not know whether or not the graph has been stretched, shrunk, or is standard width. Without a scale on the y-axis, we cannot find the value of **a** for this exact graph. So, any equation in the form $y = a(x^2 - x - 12)$ is a possible equation for the graph shown.

**Example:**

Write two possible equations for the graph shown.

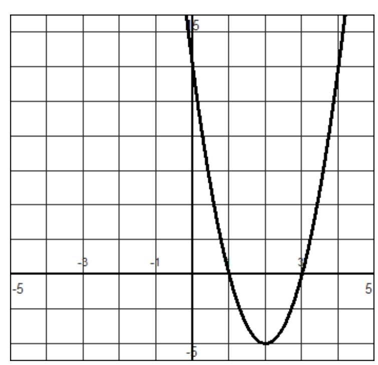


**Solution:**
The x-intercepts are (-4, 0) and (-1, 0).
$x = -4$ or $x = -1$
$x + 4 = 0$ or $x + 1 = 0$
$(x + 4)(x + 1) = 0$
$x^2 + 5x + 4 = 0$
Two possible equations are $y = x^2 + 5x + 4$ or $y = 2(x^2 + 5x + 4)$.
Note that the value of the constant **a** was chosen to be 2 but it could be any positive number.

If there is a scale on the y-axis or if we can identify another point on the graph, we can find a value of **a** which gives the equation of the graph. Using the third point, plug the values of x and y into the equation and solve for **a**.

**Examples:**

1. Find the equation for the graph shown.




**Solution:**
The x-intercepts are (1, 0) and (3, 0).
$x = 1$ or $x = 3$
$(x - 1)(x - 3) = 0$
$y = a(x^2 - 4x + 3)$
Using the point (0, 12):
$12 = a(0^2 - 4(0) + 3)$ or $12 = 3a$
$a = 4$ so the equation is $y = 4(x^2 - 4x + 3)$.

2. Find an equation for a graph which has intercepts at (6, 0), (9, 0) and (0, -2).

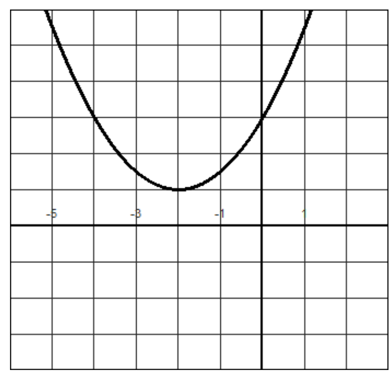
**Solution:**
Using the x-intercepts:
x = 6 or x = 9
$(x - 6)(x - 9) = 0$
$x^2 - 15x + 54 = 0$
So the equation must have the form $y = a(x^2 - 15x + 54)$. Plugging in the y-intercept gives:
$-2 = a (54)$
$a = -2/54 = -1/27$.
The equation is $y = -1/27(x^2 - 15x + 54)$.

**Writing Quadratic Functions in Vertex Form**

Previously we learned how to write the equation of a quadratic in vertex form given the graph.

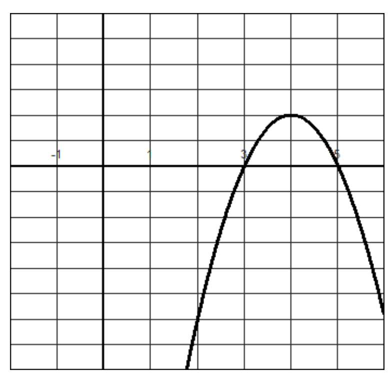
**Examples:**

1. Write the equation of the graph shown.



**Solution:**
The vertex of the graph is (-2, 1). The two points which are one unit to the right and one unit to the left of the vertex are ½ unit up.
Therefore, the equation is $f(x) = ½ (x + 2)^2 + 1$.

2. Write the equation of the graph shown.

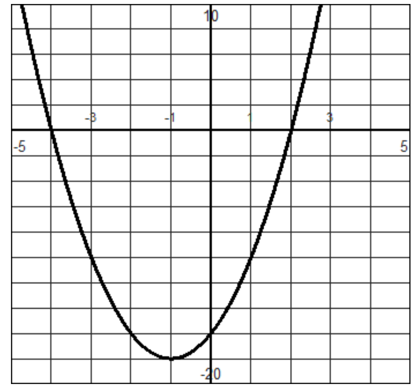


**Solution:**
The vertex of the graph is (4, 2). The two points which are one unit to the right and one unit to the left of the vertex are 2 units down.
Therefore, the equation is $f(x) = -2 (x - 4)^2 + 2$.

Let's now compare the two methods we have used previously using the same graph.

**Example:**

Given the graph shown, write the equation in A) vertex form and B) standard form.



**Solution:**

A) We can see from the graph that the vertex is (-1, -18). Therefore, in vertex form the equation will be $y = a(x + 1)^2 - 18$. Choosing another point, we can find the value of **a**. Let's choose (0, -16).

$-16 = a\,(0+1)^2 - 18$
$-16 = a - 18$
$a = 2$ so the equation is $y = 2(x + 1)^2 - 18$.

B) We can see from the graph that the x-intercepts are (0, -4) and (0, 2). Working backwards:

$x = -4$ and $x = 2$
$(x+ 4)(x - 2) = 0$
$X^2 + 2x - 8 = 0$
So the equation is $y = a(x^2 + 2x - 8)$. Choosing any other point, we can find the value of **a**. Let's choose (0, -16).

$-16 = a(-8)$ or $a = 2$. So the equation is $y = 2(x^2 + 2x - 8)$.

**Writing a quadratic equation through any three points**

Previously, we gathered data and used linear regression to find a line that approximated the data. We can find a quadratic function that approximates data if the data looks like it is quadratic in shape. We can use the elimination method to find a quadratic equation through any three points. The equation will be in standard form.

**Steps to write the equation:**

1. Identify any three points on the graph.
2. Use the equation $y = ax^2 + bx + c$ and the three points to write a system of three equations by plugging in the values of x and y for each of the three points.
3. Use the elimination method to eliminate **c** using 2 different pairs of equations.
4. Solve the new system of two equations for either **a** or **b** using elimination. Find the other variable.
5. Find the value for **c** by plugging the values for **a** and **b** into any of the three original equations.
6. Plug the values for **a**, **b**, and **c** into the standard form of the equation.
7. Check.

**Example:**

Write an equation of a quadratic through the points (1, 7), (3, 35) and (-2, 25).

**Solution:**

Using $y = ax^2 + bx + c$ and plugging in each of the three points:
$7 = a(1)^2 + b(1) + c$
$35 = a(3)^2 + b(3) + c$
$25 = a(-2)^2 + b(-2) + c$

This gives the three equations:
(1) $a + b + c = 7$
(2) $9a + 3b + c = 35$
(3) $4a – 2b + c = 25$

Choosing 2 pairs of equations, eliminate **c**:
We can multiply equation (1) by -1 and add to equation (2):
(1) $-a – b – c = -7$
(2) $\underline{9a + 3b + c = 35}$
(4) $8a + 2b = 28$

We can multiply equation (1) by -1 and add to equation (3):
(1) $-a – b – c = -7$
(3) $\underline{4a – 2b + c = 25}$
(5) $3a – 3b = 18$

Using equations (4) and (5), solve for **a** and **b**.
We can multiply equation (4) by 3 and equation (5) by 2 to eliminate **b**.
(4) $24a + 6b = 84$
(5) $\underline{6a – 6b = 36}$
$30a = 120$
$a = 4$

Plugging back into equation (4) to find **b**:
(4) $8(4) + 2b = 28$
$32 + 2b = 28$
$2b = -4$
$b = -2$

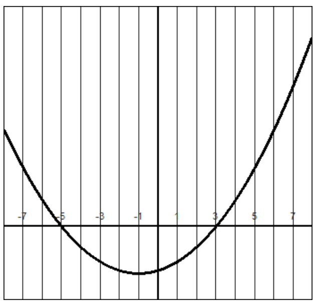
Plugging back into equation (1) to find **c**:
(1) $4 + (-2) + c = 7$
$c = 5$

The equation of the quadratic function is $y = 4x^2 - 2x + 5$. Verify that this equation is satisfied by the three given points.

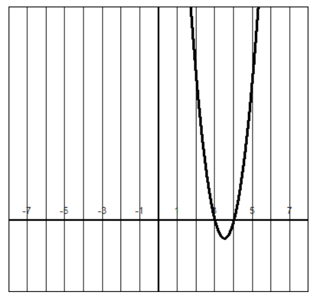
**2.5 Homework:**

Write 2 possible equations for the graphs shown.

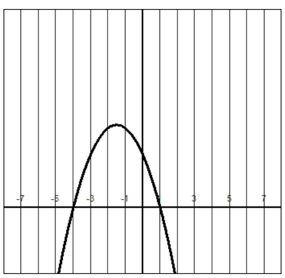
1.



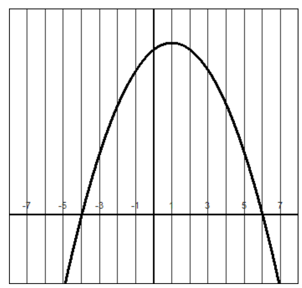

2.




3.




4.




5. Write the equation for a graph with the points (4, 0), (-7, 0) and (1, 3).

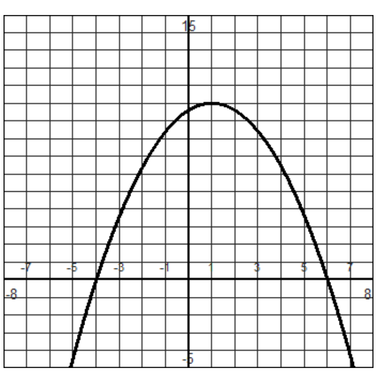
6. Write the equation for a graph with the points (-5, 0), (-2, 0) and (0, 3).

7. Write the equation for a graph with the points (-6, 0), (7, 0) and (0, -4).

8. Write the equation for a graph with the points (1/2, 0), (-3, 0) and (4, 5).

Write equations for the graphs shown using the intercepts.

9.

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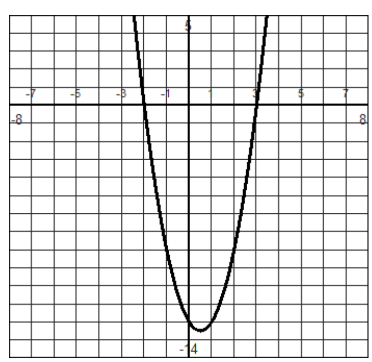

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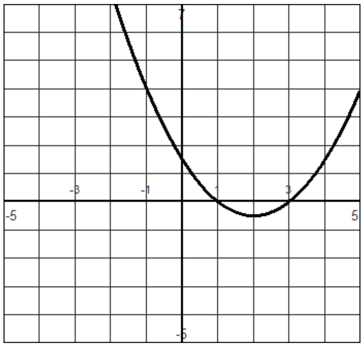

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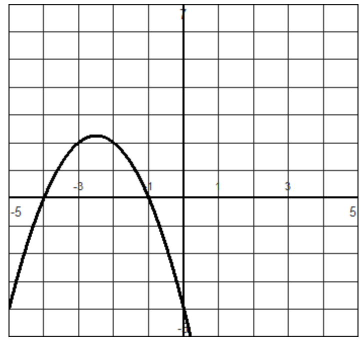
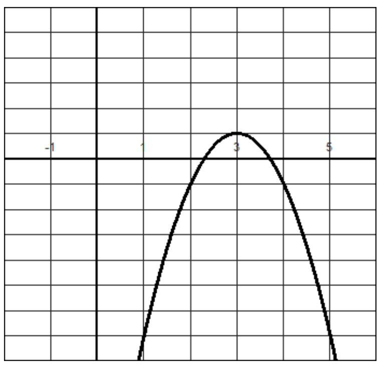


Write the equation of the graph shown in vertex form.

13.



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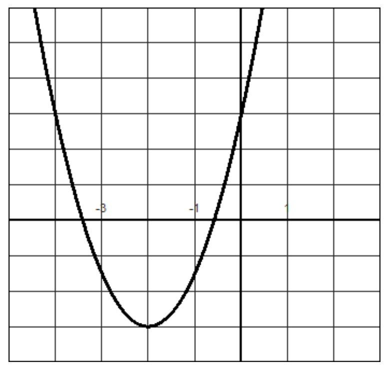

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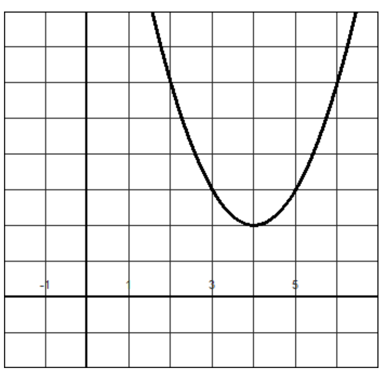

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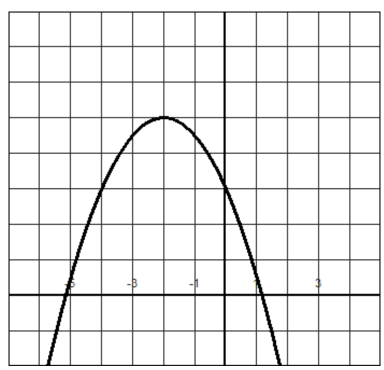
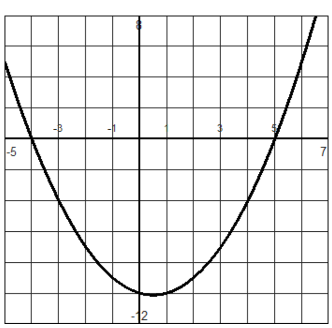
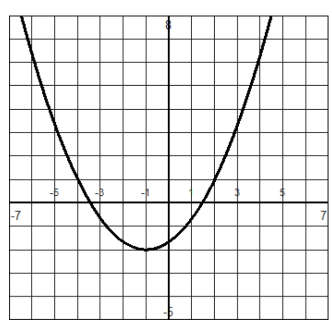


Find equations of the following graphs.

17.



18.



19. Write the equation of the parabola that has a vertex at (1/2, -5) and the point (2, 7).

20. Write the equation of the parabola that has a vertex at (-3, -5) and the point (1, -8).

21. Write the equation of the parabola that has the points (3, 0), (5, 0) and (0, 10).

22. Write the equation of the parabola that has the points (-2, 0), (4, 0) and (-1, 2).

23. Write the equation of the parabola that has the points (-2, 14), (2, 10), and (3, 19).

24. Write the equation of the parabola that has the points (-4, 5), (2, 7), and (6, 35).

25. Write the equation of the parabola that has the points (-3, -13), (-1, 0), and (1, 1).

26. At 1821 feet tall, the CN Tower in Toronto, Ontario, is the world's tallest self-supporting structure. Suppose you are standing in the observation deck on top of the tower and you drop a penny from there and watch it fall to the ground. The table shows the penny's distance from the ground after various periods of time (in seconds) have passed. Write an equation for the height of the penny at any time t.

| **Time (seconds)** | **Distance (feet)** |
|---|---|
| 0 | 1821 |
| 2 | 1757 |
| 6 | 1245 |
| 10 | 221 |

27. The table lists the number of Americans (in thousands) who are expected to be over 100 years old for selected years. [Source: US Census Bureau.] Write an equation for the number of Americans in thousands who are expected to be over 100 years old where t is time in years after 1990. How many Americans will be over 100 years old in the year 2008?

| Year | Number (thousands) |
|---|---|
| 1994 | 50 |
| 1998 | 65 |
| 2000 | 75 |
| 2002 | 94 |
| 2004 | 110 |

28. The table shows the cost of driving a car at different speeds. The speeds, V, are given in miles per hour and the cost, C, includes fuel and maintenance for driving the car 100 miles at that speed.

| | | | | | |
|---|---|---|---|---|---|
| v | 30 | 40 | 50 | 60 | 70 |
| C | 6.5 | 6 | 6.2 | 7.8 | 10.6 |

29. The batter in a softball game hits the ball when it is 3.5 feet above the ground. The ball reaches its highest trajectory of 38 feet when it is 150 feet from home plate. Write an equation for the height of the softball in terms of its horizontal distance from home plate.

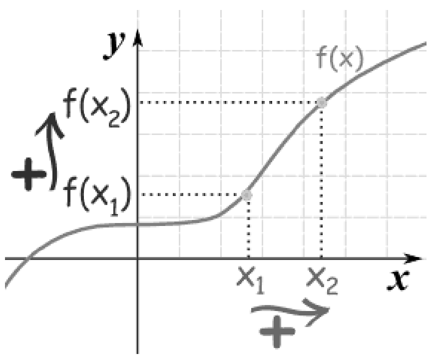
30. An acrobat is catapulted into the air from a springboard at ground level. Her maximum height of 11.025 meters is achieved after 1.5 seconds. Write an equation for the path of the acrobat.

31. Sam is standing 30 feet away from a basket and shoots the ball at the basket from a height of 6.5 feet. The hoop is 10 feet high. If the ball reaches a maximum height of 11.5 feet when it is 1 foot away from the basket, write an equation for the trajectory of the ball.

## 2.6 Increasing/Decreasing, Maximum/Minimum, and Applications of Quadratics

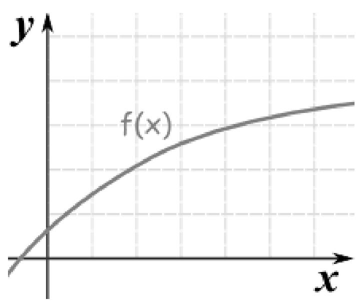
### Increasing and Decreasing Functions

A function is increasing if the **outputs** increase as the **inputs** increase, like this:



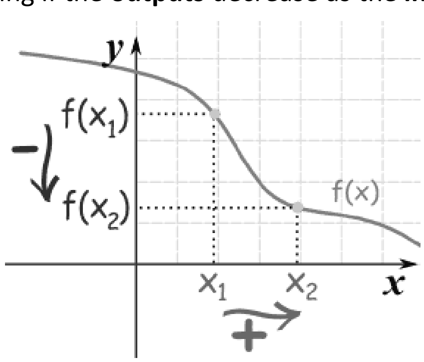

We can see that an increasing graph will rise from left to right. Mathematically, f(x) is increasing if for every $x_1 < x_2$, then $f(x_1) < f(x_2)$. This has to be true for **any** $x_1$, $x_2$, not just some input values. The rate of increase does not have to be constant.

**Example:**



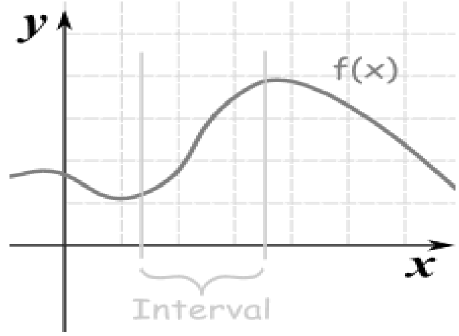

This is an increasing function whose rate of increase slows from left to right.

A function is decreasing if the **outputs** decrease as the **inputs** increase, like this:




We can see that a decreasing graph will fall from left to right. Mathematically, f(x) is decreasing if for every $x_1 < x_2$, then $f(x_1) > f(x_2)$. This has to be true for **any** $x_1$, $x_2$, not just some input values. The rate of decrease does not have to be constant.

Usually we will be interested in **some interval**, like this one which is increasing. The function is increasing on the interval shown (it may be increasing or decreasing elsewhere).




For a quadratic function, the graph changes from increasing to decreasing (for a concave down parabola) or from decreasing to increasing (for a concave up parabola) at the vertex. The vertex is called a **turning point** of the graph. For other graphs, the turning points may not be as easy to locate. These turning points occur at either maximum or minimum values of the function. There are two types of maximum and minimum values. These are called **global maximum or minimum values** when the values are the absolute highest or lowest on the entire domain of the function. If the maximum or minimum value is just higher or lower than the surrounding points on the graph, then these are called **local maximum or minimum values**.

**Examples:**

1. For $f(x) = 2(x - 3)^2 + 5$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

   **Solution:**

   This graph is concave up because the value of a is positive. Since the equation is in vertex form, we can see that the vertex is at (3, 5). The vertex will be the lowest point on the graph; therefore, the minimum value is 5. The graph will be decreasing on the interval $(-\infty, 3)$ and increasing on the interval $(3, \infty)$.

2. For $f(x) = -x^2 + 4x – 7$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

   **Solution:**

   This graph is concave down because the value of a is negative. We can find the vertex by using the formula $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$. Plugging x=2 back into the equation, we find the vertex to be (2, -3). The vertex will be the highest point on the graph; therefore, themaximum value is -3. The graph will be increasing on the interval $(-\infty, 2)$ and decreasing on the interval $(2, \infty)$.

Many of our applications in this chapter will revolve around minimum and maximum values of quadratic functions. Quadratic equations can appear in applications such as revenue, stopping distance of a car, height of a dropped or thrown object, and area problems.

When setting up the equation you should:

1. Write down given information. Determine what the variable represents.
2. Sketch a diagram if appropriate.
3. Identify any needed formulas.
4. Write the equation.
5. Solve using factoring, taking square roots, or the quadratic formula.
6. Verify the solutions and check the suitability of the solutions.

**Examples:**

1. We are going to fence in a rectangular field and we know that for some reason we want the field to have an enclosed area of 75 $ft^2$. We also know that we want the width of the field to be 3 feet longer than the length of the field. What are the dimensions of the field?

   **Solution:**

   If we let L be the length of the field, then we know that the width which is 3 feet longer than the length must be L+3. We are given that the area is 75 $ft^2$.

   Next, we know the area of a rectangle is its length times width.

   $A = L \cdot W$
   $75 = L\,(L + 3)$
   $75 = L^2 + 3L$
   $L^2 + 3L - 75 = 0$

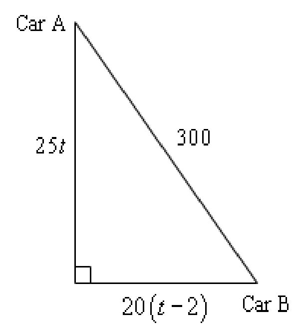
   Using the quadratic formula, we find L = $\frac{-3 \pm \sqrt{309}}{2}$. In decimal form, the two solutions to this equation are L = 7.289 and L = -10.289. As the length of a rectangle cannot be negative, the only reasonable solution is L = 7.289. Therefore, the length is 7.289 feet and the width will be 10.289 feet.

2. Two cars start out at the same point. One car starts out driving north at 25 mph. Two hours later, the second car starts driving east at 20 mph. How long after the first car starts traveling does it take for the two cars to be 300 miles apart?

**Solution:**

We'll start off by letting *t* be the amount of time that the first car, let's call it car A, travels. Since the second car, let's call that car B, starts out two hours later then we know that it will travel for t-2 hours.

Now, we know that the distance traveled by an object is its rate of travel (speed) times time traveled. So we have the following distances traveled for each car.
Distance of car A: 25t
Distance of car B: 20(t – 2)




Sketching a picture of the situation, we can see that we have a right triangle. That means that we can use the Pythagorean Theorem.

$(25t)^2 + (20(t - 2))^2 = 300^2$

Multiplying and simplifying gives:

$1025t^2 – 1600t – 88400 = 0$

Using the quadratic formula, we find t = $\frac{1600 \pm \sqrt{365000000}}{2050}$. Estimating the solutions, we get t = 10.1 and t = -8.54. The negative solution does not make sense because we are discussing time. Therefore, the time after the first car starts traveling for the two cars to be 300 miles apart is 10.1 hours.

3. The height,h , in feet of an object above the ground is given by $h = -16t^2 + 64t + 190$, where *t* is the time in seconds. Find the time it takes the object to strike the ground and find the maximum height of the object.

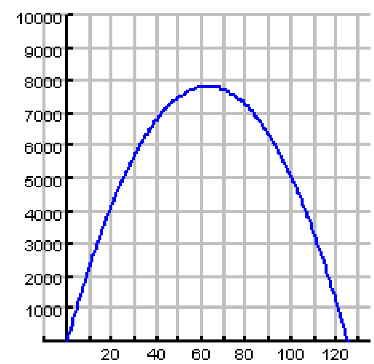
**Solution:**

The ground is 0 feet high. So, we need to solve $-16t^2 + 64t + 190 = 0$.
Using the quadratic formula we find t = $\frac{8 \pm \sqrt{254}}{4}$ or t = 5.98 and t = -1.98. As negative time is not reasonable, the time it takes for the object to hit the ground is 5.98 seconds.

The other part of the question is we want to know that maximum height that the object reaches. Since we can see that the function is clearly a quadratic function which opens down, we know that this maximum must occur at the vertex. So let's find the vertex. We use the formula $t = \frac{-b}{2a}$. So we have t = $\frac{-64}{2(16)}$ = 2 seconds.

So at 2 seconds the object reaches its maximum height. However, we wanted to know what that maximum height is. Therefore, we must find the value of the vertex, in this case it will be the value of h when t = 2. So we plug this in to the equation and find h = 254 feet. So the maximum height is 254 feet.

4. A parking lot is rectangular in shape and measures 250 feet around three of the four sides. Assuming the width of the parking lot is x, then the area of the lot can be modeled by A = (250-2x)x. A graph of the function is shown to the right. For what widths is the area increasing? For what widths is the area decreasing? For what width does the lot have its maximum area?




**Solution:**

As the area $A = 250x - 2x^2$, we can find that the vertex is at (62.5, 7812.5) using the vertex formula. The area of the parking lot is increasing on the interval [0, 62.5) and decreasing on the interval (62.5, 125]. Notice that the domain of this function is [0, 125]. The lot will have its maximum area of 7812.5 feet when the width of the lot is 62.5 feet.

**2.6 Homework:**

1. For $f(x) = -(x - 7)^2 + 5$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

2. For $f(x) = (x + 11)^2 - 8$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

3. For $f(x) = 3x^2 - 5x + 10$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

4. For $f(x) = -2x^2 - 7x + 1$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

5. We are going to fence in a rectangular field and we know that we want the field to have an enclosed area of 90 $ft^2$. We also know that we want the width of the field to be 5 feet longer than the length of the field. What are the dimensions of the field?

6. Two cars start out at the same point. One car starts out driving south at 35 mph. Three hours later the second car starts driving east at 25 mph. How long after the first car starts traveling does it take for the two cars to be 400 miles apart?

7. An object is dropped off of a 275-foot building and its height at time t, in seconds, is given by $h = -16t^2 + 275$, where h is in feet.
   A. How long will it take the object to hit the ground?
   B. At what times is the height of the object increasing?
   C. At what times is the height of the object decreasing?

8. A ball is thrown across a field. The height of the ball at time t, in seconds, is given by $h(t) = -4.9t^2 + 15t + 2$ where h is in meters.
   A. If someone catches the ball at a height of 3 meters on its way down, how long was the ball in the air?
   B. Find the time it takes the ball to hit the ground if it is not caught.
   C. At what times was the height of the ball increasing?
   D. When does the ball reach its maximum height? What is the maximum height?

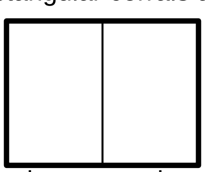
9. The number of bacteria in a refrigerated food is given by$N(t) = 20T^2 – 20T + 120$, for $-2 \le T \le 14$ and where *T* is the temperature of the food in Celsius. At what temperature will the number of bacteria be minimal?

10. An object is dropped off of a 300-foot building and its height at time t, in seconds, is given by $h = -16t^2 + 300$, where h is in feet. How long will it take the object to hit the ground?

11. The height,h , in feet of an object above the ground is given by $h = -16t^2 + 48t + 150$, where *t* is the time in seconds. Find the time it takes the object to strike the ground and find the maximum height of the object.

12. The length of a rectangle is three more than twice the width.
    A. What is the minimum area that this rectangle can have?
    B. Determine the dimensions that will give a total area of 27 $m^2$.

13. Two rectangular corrals are to be made from 100 yds of fencing as seen below.



    If the rancher wants the total area to be maximum, what dimensions should be used to make the corrals?

14. The number of board feet in a 16 foot long tree is approximated by the model $F(d) = 0.77d^2 -1.32d – 9.31$ where *F* is the number of feet and *d* is the diameter of the log in inches.
    A. How many board feet are in a log with diameter 12 inches?
    B. What is the diameter that will produce the minimum number of board feet?

15. The number of horsepower needed to overcome a wind drag on a certain automobile is given by $H(s) = 0.05s^2 + 0.007s – 0.031$, where *s* is the speed of the car in miles per hour.
    A. How much horsepower is needed to overcome the wind drag on this car if it is traveling 50 miles per hour?
    B. At what speed will the car need to use 200 horsepower to overcome the wind drag?

16. A manufacturer of tennis balls has a daily cost of $C(x) = 200 – 10x + 0.01x^2$ where *C* is the total cost in dollars and *x* is the number of tennis balls produced.
    A. What number of tennis balls will produce the minimum total cost? What is the minimum cost?
    B. On what interval will the total cost be decreasing?

17. A textile manufacturer has daily production costs of $C(x) = 10000 – 110x + 0.05x^2$, where is the total cost (in dollars) and is the number of units produced. How many units should be produced each day to yield a minimum cost?

18. A company earns a weekly profit of $P(x) = -0.75x^2 + 60x – 300$ dollars by selling x items. How many items does the company have to sell each week to maximize the profit?

19. A ball rolls down a slope and travels a distance $d = 6t + ½ t^2$ feet in *t* seconds. Find when the distance is 17 feet.

20. The video screen in the Dallas Cowboys stadium is 11520 square feet. The length is 88 feet more than the width. Find the dimensions of the video screen.

21. The path of a high diver is given by $y = -\frac{4}{9}x^2 + \frac{24}{9}x + 10$ where y is the height in feet above the water and x is the horizontal distance from the end of the diving board in feet. What is the maximum height of the diver and how far out from the end of the diving board is the diver when he hits the water?

22. The length of a rectangular plot of land is 10 yards more than its width. If the area of the land is 600 square yards, find the dimensions of the plot of land.

23. The height of a triangular window is 3 feet less than its base. If the area of the window is 20 square feet, find the dimensions of the window.

24. The length of a Ping-Pong table is 3 ft more than twice the width. The area of a Ping-Pong table is 90 square feet. What are the dimensions of a Ping-Pong table?

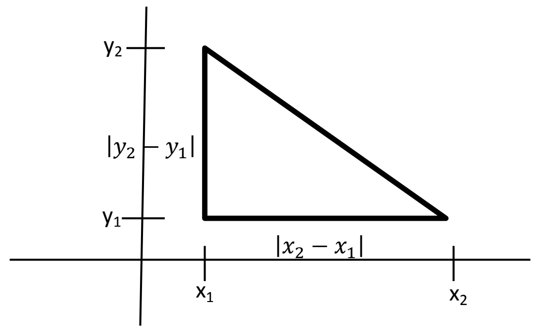
25. Three hundred feet of fencing is available to enclose a rectangular yard along side of the St. John's River, which is one side of the rectangle.
    A. What dimensions will produce an area of $10,000 ft^2$?
    B. What is the maximum area that can be enclosed?

23. Five hundred feet of fencing is available to enclose a rectangular lot along side of highway 65. Cal Trans will supply the fencing for the side along the highway, so only three sides are needed.
    A. What dimensions will produce an area of $40,000 ft^2$?
    B. What is the maximum area that can be enclosed?

## 2.7 Distance and Midpoints

### The Distance Formula

The distance between two points in the Cartesian coordinate system can be calculated using the Pythagorean Theorem. Recall that the Pythagorean Theorem states that for a right triangle with legs of lengths a and b and a hypotenuse of length c then $a^2 + b^2 = c^2$. Look at the triangle shown below.



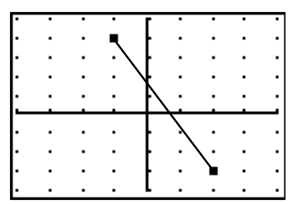

The lengths of the legs are $|x_2 - x_1|$ and $|y_2 - y_1|$. The distance between the points $(x_1, y_1)$ and $(x_2, y_2)$ is the hypotenuse of the triangle. Applying the Pythagorean Theorem, $d^2 = (x_2 - x_1)^2 - (y_2 - y_1)^2$. Since distance is non-negative, then the distance, d, between the points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

**Examples:**

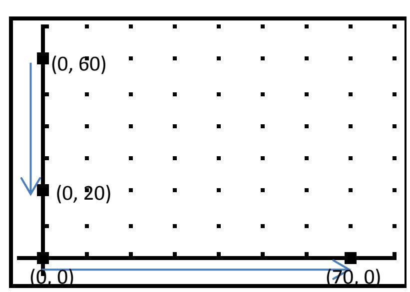
1. Find the length of the line segment between the points (-1, 4) and (2, -3).

$$d = \sqrt{(2-(-1))^2 + (-3-4)^2} = \sqrt{9+49} = \sqrt{58} \approx 7.62$$



2. Suppose that at noon car A is traveling south at 20 miles per hour and is located 60 miles north of car B. Car B is traveling east at 35 miles per hour. Let (0, 0) be the coordinates of car B where the units are in miles. Plot the location of each car at 2:00 PM and find the distance between the cars at this time.

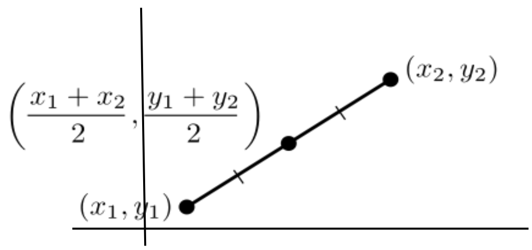
Car B started at (0, 0) and traveled at 35 miles per hour for two hours. So it traveled a distance of 70 miles. Car A started 60 miles north or at the point (0, 60) and traveled south at 20 miles per hour for two hours for a



distance of 40 miles. Thus, its new coordinate is (0, 20).
To find the distance between the two cars, we use the distance formula. d = $\sqrt{(70-0)^2 + (0-20)^2} = \sqrt{4900+400} = \sqrt{5300} \approx 72.8\ miles.$

**The Midpoint Formula**

If a line segment is drawn between two points, then its midpoint is the point on the line segment that is equidistant from the endpoints. On a number line, the midpoint between two points is the average of the two coordinates. The midpoint in the Cartesian coordinate system is similar to the formula for the midpoint on a number line except that both coordinates are averaged. Thus, the midpoint between points ($x_1$, $y_1$) and ($x_2$, $y_2$) is the point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. See the picture below.




**Example:**

If the population of the United States was 203 million in 1970 and in 1990 the population was 249 million, use the midpoint formula to estimate the population in 1980.

We are given two points with this data. The points are (1970, 203) and (1990, 249). The midpoint of these two points is:

$$\left(\frac{1970+1990}{2}, \frac{203+249}{2}\right) = (1980, 226)$$

Therefore, the estimated population is 1980 is 226 million.

**2.7 Homework:**

1. Find the distance between the points (-4, 5) and (0, 7).
2. Find the distance between the points (1, -3) and (-2, 6).
3. Find the distance between the points (2.5, 3) and (1.75, 3.2).
4. Find the distance between the points (-0.1, 8.1) and (6.3, 9.2).
5. An isosceles triangle has at least two sides of equal length. Determine whether the triangle with vertices (0, 0), (3, 4) and (7, 1) is isosceles.

6. Find the midpoint between the points (9, 4) and (-1, 2).

7. Find the midpoint between the points (-2, 7) and (-6, -2).

8. At 10:00 AM, car A is traveling north at 40 miles per hour and is located 50 miles south of car B. Car B is traveling west at 30 miles per hour.

   A. Let (0, 0) be the initial coordinates of car B, where the units are in miles. Plot the locations of each car at 10:00 AM and 12:30AM.

   B. Find the distance between the cars at 12:30 AM.

   C. Find the midpoint between the two cars at 12:30 AM.

9. In the year 1983, the average daily jail population was 228 thousand people. In 1993, the average daily jail population had risen to 466 thousand people. Find the midpoint and explain what it represents in context.

10. In the year 1987, the federal debt was 2354 billion dollars. This had risen to 2881 billion dollars in 1989. Find the midpoint and explain what it represents in context.

11. Find the equation of a circle with a center of (8, -2) which has the point (-2, 5) on the circle.

12. Find the equation of a circle which has the endpoints of a diameter of the circle at (-5, -7) and (1, 3).